# Projective Geometric Algebra and Rigid Transformations

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NASA GN&C

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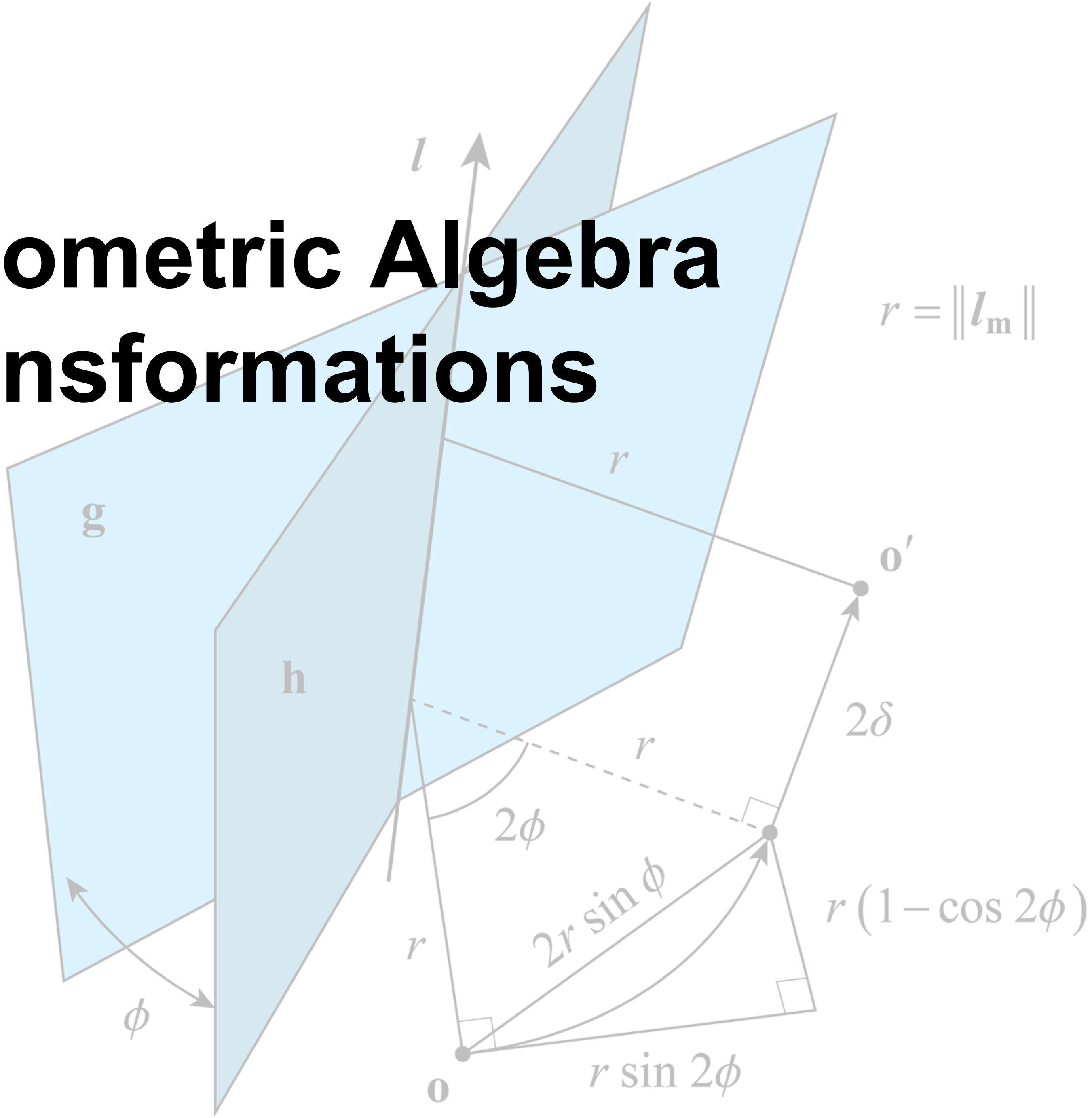
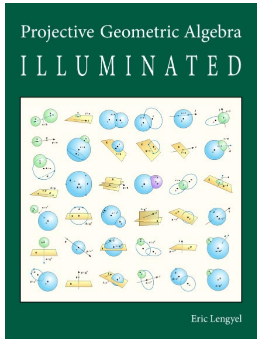


# About the Speaker

- Computer Scientist / Mathematician
- Working in industry since 1994
- Running company that specializes in digital typography and game engines
- Writing books, occasionally teaching



Projective Geometric Algebra
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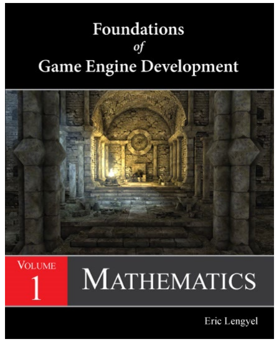

Foundations
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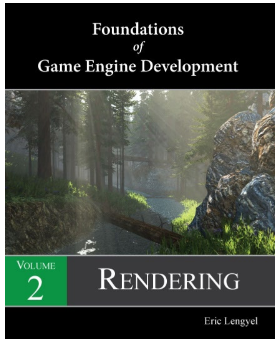

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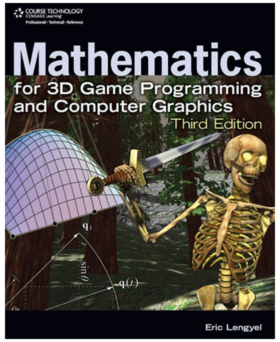

Mathematics
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and Computer Graphics
Third Edition
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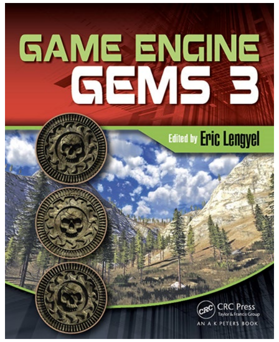

GAME ENGINE
GEMS 3
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CRC Press

SLUG

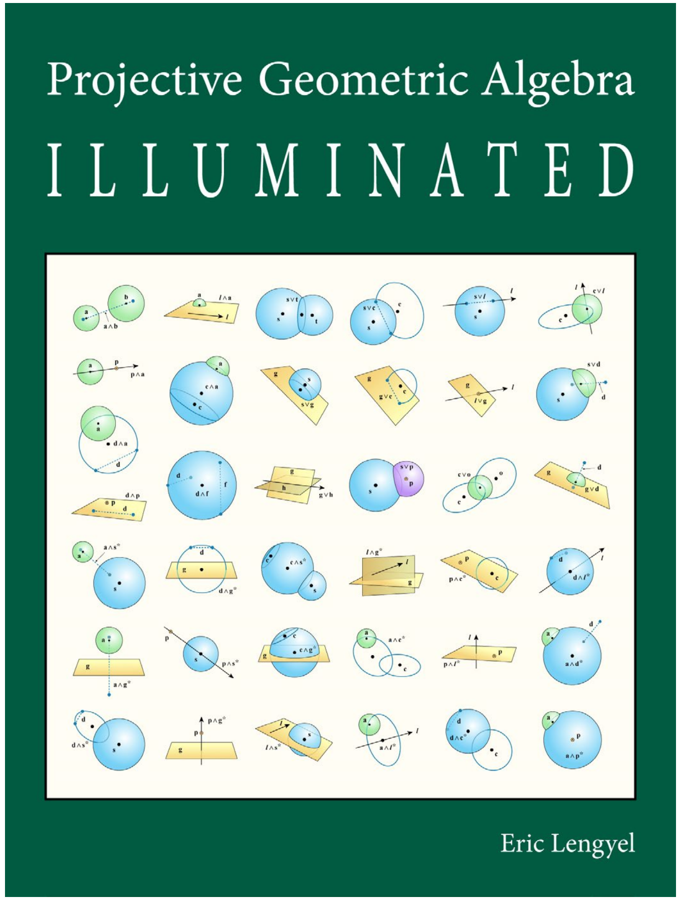
C4 ENGINE

OpenGEX

TRANSVOXEL

# Subject of This Talk

- 4D rigid exterior algebra
  - Homogeneous representation of 3D geometry
  - Points, lines, planes
  - Join, meet, projection, norm, distance, angle
- 4D rigid geometric algebra
  - Euclidean isometries in 3D space
  - Rotations, translations, screw transformations
  - Parameterization, interpolation
- Details in PGA Illuminated



Projective Geometric Algebra

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# Exterior / Grassmann Algebra

- Wedge product ∧
  - Combines dimensions of operands
  - Vectors square to zero:

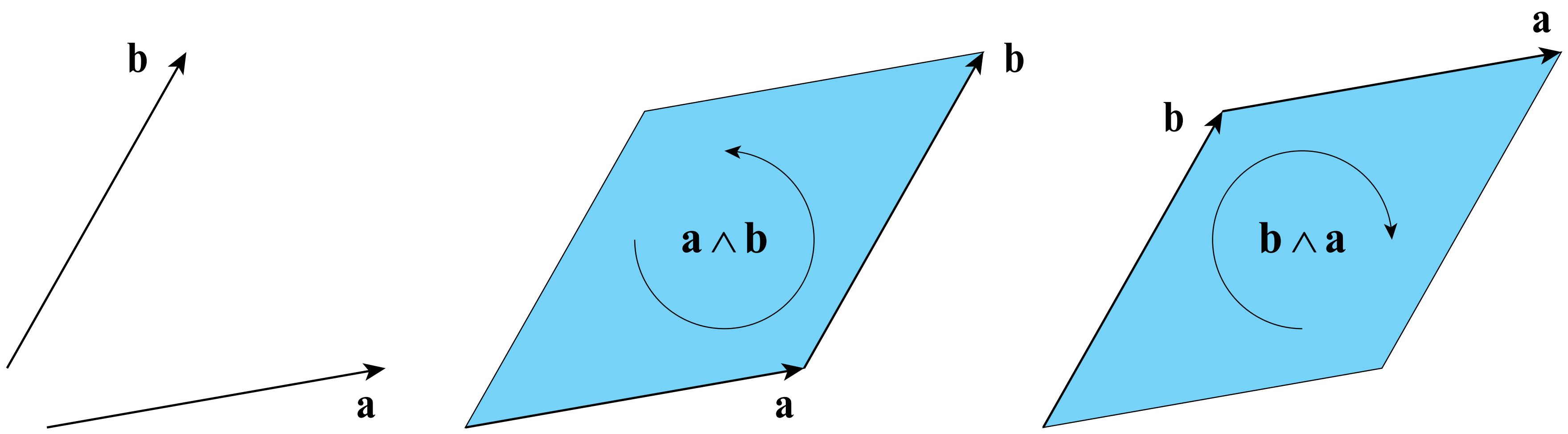
$$\mathbf{v} \wedge \mathbf{v} = 0$$

  - Antisymmetric on vectors:

$$\mathbf{a} \wedge \mathbf{b} = -\mathbf{b} \wedge \mathbf{a}$$

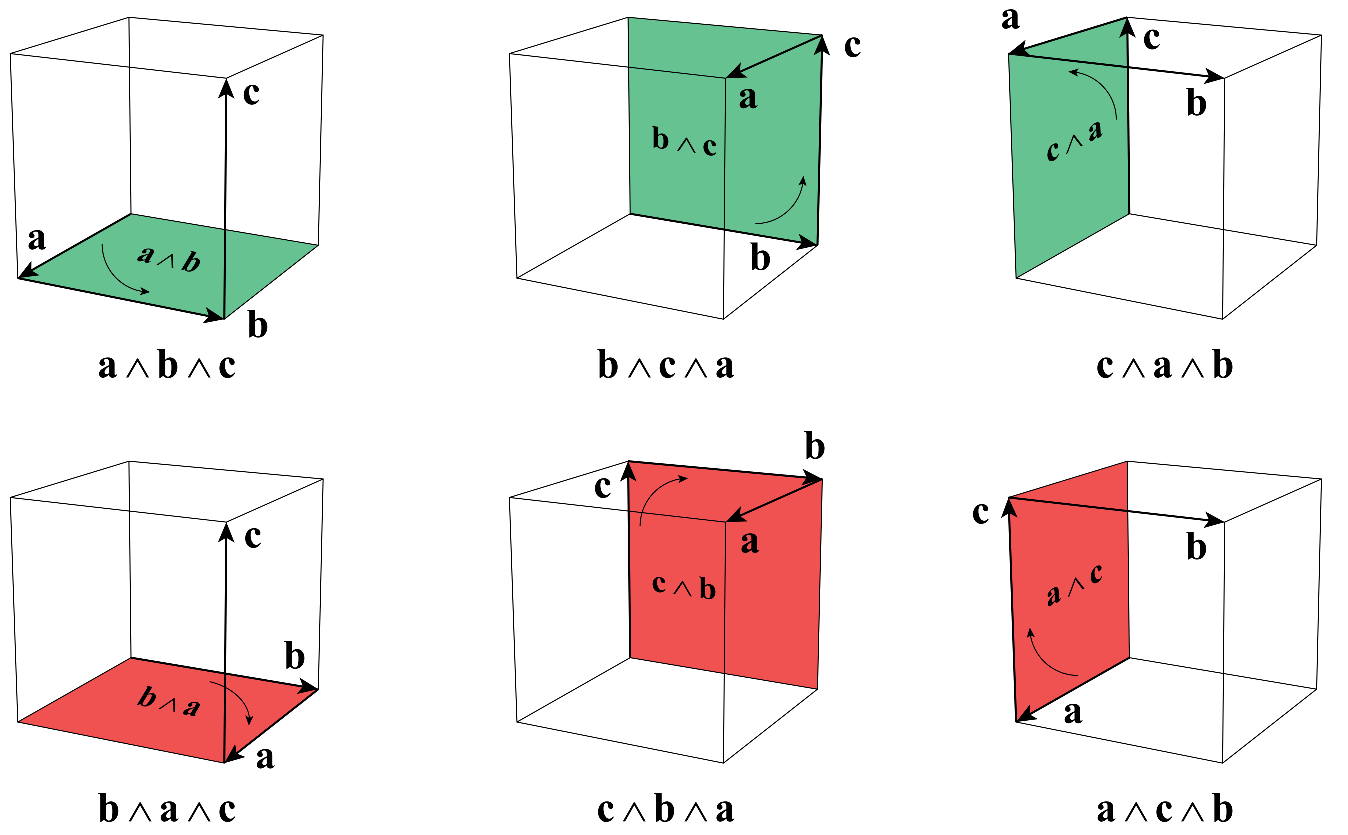
# Bivectors

- Wedge product of two vectors **a** and **b**

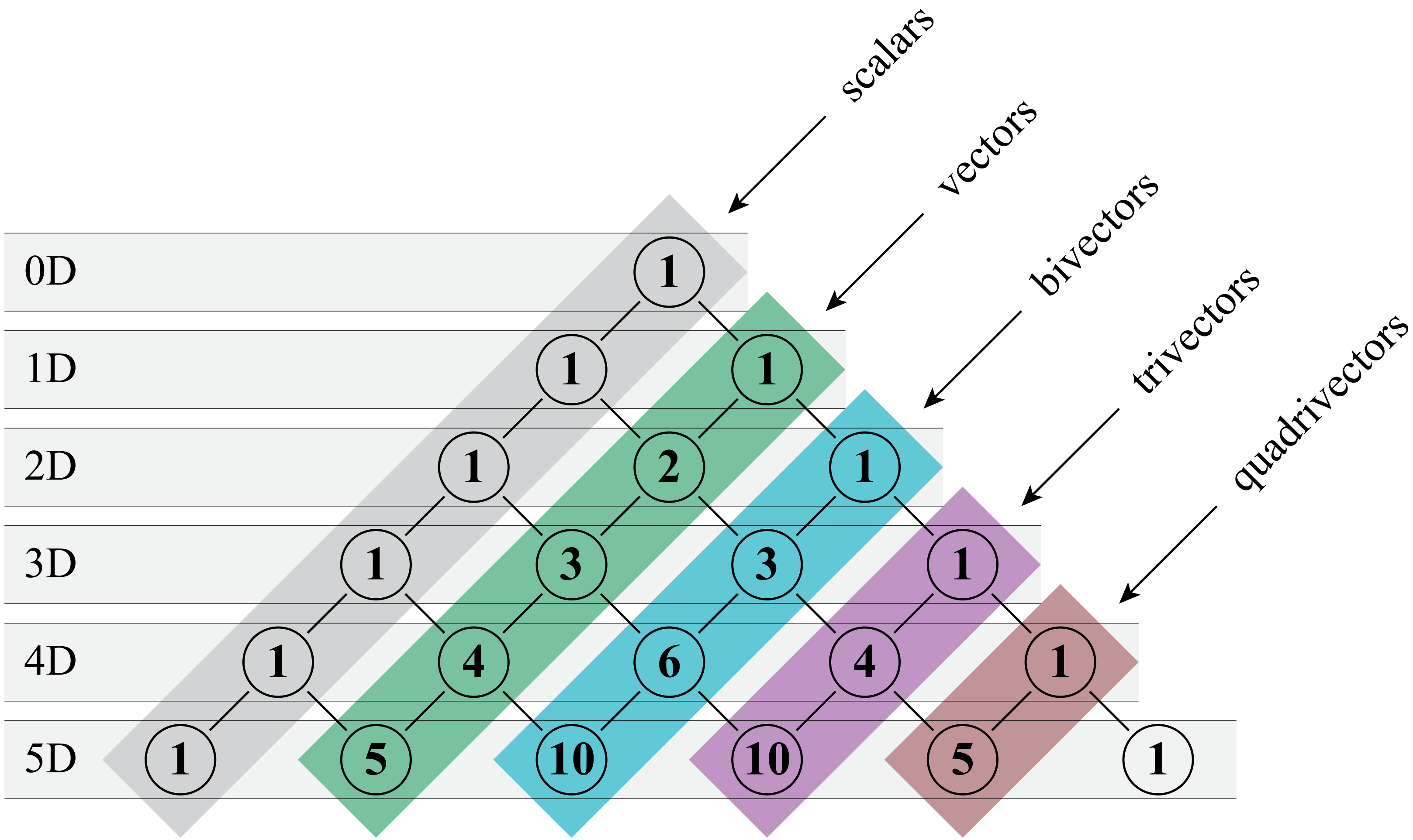


# Trivectors

- Wedge product of three vectors **a**, **b**, and **c**



# Pascal’s Triangle



# Rigid Exterior / Geometric Algebra

- Projective algebra with one extra dimension
- Contains points, lines, planes in 3D
- Can perform rotations, translations, screw transformations

# 4D Exterior Algebra

- Extends 4D vector space

- One scalar **1**
- Four vector basis elements
- Six bivector basis elements
- Four trivector basis elements
- One antiscalar $\mathbb{1}$

| Type | Values | Grade / Antigrade | |
|---|---|---|---|
| Scalar | $\mathbf{1}$ | 0 / 4 | ▯▯▯▯ |
| Vectors | $\mathbf{e}_1$<br>$\mathbf{e}_2$<br>$\mathbf{e}_3$<br>$\mathbf{e}_4 = \mathbf{e}_n$ | 1 / 3 | ▮▯▯▯<br>▯▮▯▯<br>▯▯▮▯<br>▯▯▯▮ |
| Bivectors | $\mathbf{e}_{41} = \mathbf{e}_4 \wedge \mathbf{e}_1$<br>$\mathbf{e}_{42} = \mathbf{e}_4 \wedge \mathbf{e}_2$<br>$\mathbf{e}_{43} = \mathbf{e}_4 \wedge \mathbf{e}_3$<br>$\mathbf{e}_{23} = \mathbf{e}_2 \wedge \mathbf{e}_3$<br>$\mathbf{e}_{31} = \mathbf{e}_3 \wedge \mathbf{e}_1$<br>$\mathbf{e}_{12} = \mathbf{e}_1 \wedge \mathbf{e}_2$ | 2 / 2 | ▮▯▯▮<br>▯▮▯▮<br>▯▯▮▮<br>▯▮▮▯<br>▮▯▮▯<br>▮▮▯▯ |
| Trivectors / Antivectors | $\mathbf{e}_{423} = \mathbf{e}_4 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$<br>$\mathbf{e}_{431} = \mathbf{e}_4 \wedge \mathbf{e}_3 \wedge \mathbf{e}_1$<br>$\mathbf{e}_{412} = \mathbf{e}_4 \wedge \mathbf{e}_1 \wedge \mathbf{e}_2$<br>$\mathbf{e}_{321} = \mathbf{e}_3 \wedge \mathbf{e}_2 \wedge \mathbf{e}_1$ | 3 / 1 | ▯▮▮▮<br>▮▯▮▮<br>▮▮▯▮<br>▮▮▮▯ |
| Antiscalar | $\mathbb{1} = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3 \wedge \mathbf{e}_4$ | 4 / 0 | ▮▮▮▮ |

# 4D Exterior Product

Wedge Product $\mathbf{a} \wedge \mathbf{b}$

| **a** **b** | $\mathbf{1}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
| $\mathbf{e}_{1}$ | $\mathbf{e}_{1}$ | 0 | $\mathbf{e}_{12}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{41}$ | 0 | $-\mathbf{e}_{412}$ | $\mathbf{e}_{431}$ | $-\mathbf{e}_{321}$ | 0 | 0 | $\mathbb{1}$ | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{2}$ | $\mathbf{e}_{2}$ | $-\mathbf{e}_{12}$ | 0 | $\mathbf{e}_{23}$ | $-\mathbf{e}_{42}$ | $\mathbf{e}_{412}$ | 0 | $-\mathbf{e}_{423}$ | 0 | $-\mathbf{e}_{321}$ | 0 | 0 | $\mathbb{1}$ | 0 | 0 | 0 |
| $\mathbf{e}_{3}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{31}$ | $-\mathbf{e}_{23}$ | 0 | $-\mathbf{e}_{43}$ | $-\mathbf{e}_{431}$ | $\mathbf{e}_{423}$ | 0 | 0 | 0 | $-\mathbf{e}_{321}$ | 0 | 0 | $\mathbb{1}$ | 0 | 0 |
| $\mathbf{e}_{4}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | 0 | 0 | 0 | $\mathbb{1}$ | 0 |
| $\mathbf{e}_{41}$ | $\mathbf{e}_{41}$ | 0 | $\mathbf{e}_{412}$ | $-\mathbf{e}_{431}$ | 0 | 0 | 0 | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{42}$ | $\mathbf{e}_{42}$ | $-\mathbf{e}_{412}$ | 0 | $\mathbf{e}_{423}$ | 0 | 0 | 0 | 0 | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{43}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{431}$ | $-\mathbf{e}_{423}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{23}$ | $\mathbf{e}_{23}$ | $-\mathbf{e}_{321}$ | 0 | 0 | $\mathbf{e}_{423}$ | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{31}$ | $\mathbf{e}_{31}$ | 0 | $-\mathbf{e}_{321}$ | 0 | $\mathbf{e}_{431}$ | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{12}$ | $\mathbf{e}_{12}$ | 0 | 0 | $-\mathbf{e}_{321}$ | $\mathbf{e}_{412}$ | 0 | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{423}$ | $\mathbf{e}_{423}$ | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{431}$ | $\mathbf{e}_{431}$ | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{412}$ | $\mathbf{e}_{412}$ | 0 | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbf{e}_{321}$ | $\mathbf{e}_{321}$ | 0 | 0 | 0 | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\mathbb{1}$ | $\mathbb{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Complements

- Complement inverts full / empty dimensions
- Right complement denoted by overbar
- Left complement denoted by underbar
- For basis element **u**,

$$\mathbf{u} \wedge \overline{\mathbf{u}} = \mathbb{1} \qquad \underline{\mathbf{u}} \wedge \mathbf{u} = \mathbb{1}$$

| $\mathbf{u}$ | $\mathbf{1}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_4$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\overline{\mathbf{u}}$ | $\mathbb{1}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $-\mathbf{e}_1$ | $-\mathbf{e}_2$ | $-\mathbf{e}_3$ | $-\mathbf{e}_4$ | $\mathbf{1}$ |
| $\underline{\mathbf{u}}$ | $\mathbb{1}$ | $-\mathbf{e}_{423}$ | $-\mathbf{e}_{431}$ | $-\mathbf{e}_{412}$ | $-\mathbf{e}_{321}$ | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_4$ | $\mathbf{1}$ |

# Antiproducts

- Antiwedge product denoted by ∨

- Wedge product combines dimensions that are *present*
  - Adds grades

- Antiwedge product combines dimensions that are *absent*
  - Adds antigrades

# De Morgan Laws

- Every operation with “anti” in name satisfies a De Morgan law:

$$\overline{\mathbf{a} \vee \mathbf{b}} = \overline{\mathbf{a}} \wedge \overline{\mathbf{b}} \qquad \underline{\mathbf{a} \vee \mathbf{b}} = \underline{\mathbf{a}} \wedge \underline{\mathbf{b}}$$

- To calculate anti-operation,
  - Take a complement of each input
  - Perform the regular operation
  - Take opposite complement of the result

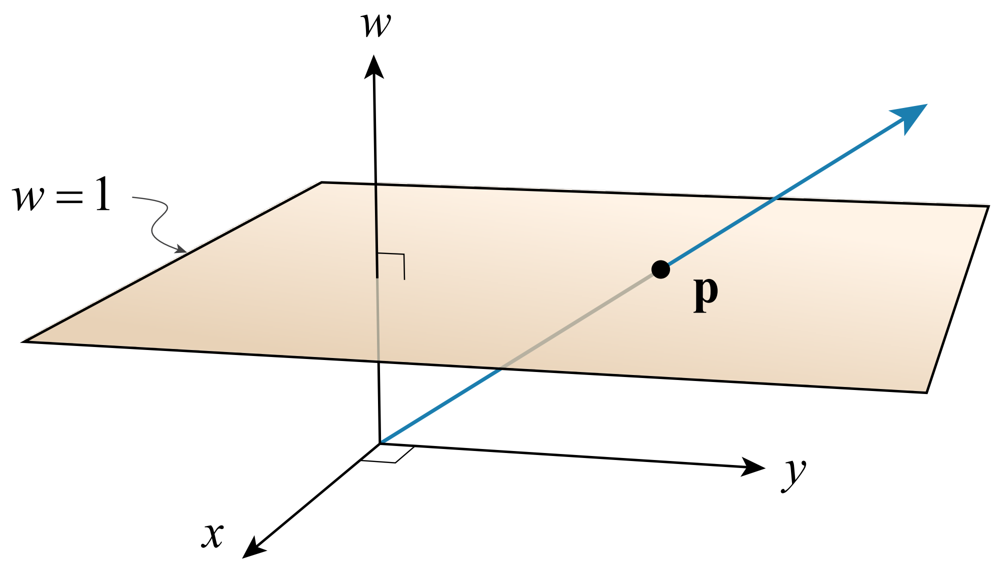
# 4D Exterior Antiproduct

Antiwedge Product $\mathbf{a} \vee \mathbf{b}$

| $\mathbf{a}$ \ $\mathbf{b}$ | $\mathbf{1}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_4$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbf{1}$ |
| $\mathbf{e}_1$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbf{1}$ | 0 | 0 | 0 | $\mathbf{e}_1$ |
| $\mathbf{e}_2$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbf{1}$ | 0 | 0 | $\mathbf{e}_2$ |
| $\mathbf{e}_3$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbf{1}$ | 0 | $\mathbf{e}_3$ |
| $\mathbf{e}_4$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbf{1}$ | $\mathbf{e}_4$ |
| $\mathbf{e}_{41}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | $-\mathbf{e}_4$ | 0 | 0 | $\mathbf{e}_1$ | $\mathbf{e}_{41}$ |
| $\mathbf{e}_{42}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | $-\mathbf{e}_4$ | 0 | $\mathbf{e}_2$ | $\mathbf{e}_{42}$ |
| $\mathbf{e}_{43}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | $-\mathbf{e}_4$ | $\mathbf{e}_3$ | $\mathbf{e}_{43}$ |
| $\mathbf{e}_{23}$ | 0 | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | 0 | 0 | 0 | 0 | $\mathbf{e}_3$ | $-\mathbf{e}_2$ | 0 | $\mathbf{e}_{23}$ |
| $\mathbf{e}_{31}$ | 0 | 0 | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | 0 | 0 | $-\mathbf{e}_3$ | 0 | $\mathbf{e}_1$ | 0 | $\mathbf{e}_{31}$ |
| $\mathbf{e}_{12}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | 0 | $\mathbf{e}_2$ | $-\mathbf{e}_1$ | 0 | 0 | $\mathbf{e}_{12}$ |
| $\mathbf{e}_{423}$ | 0 | $-\mathbf{1}$ | 0 | 0 | 0 | $-\mathbf{e}_4$ | 0 | 0 | 0 | $-\mathbf{e}_3$ | $\mathbf{e}_2$ | 0 | $-\mathbf{e}_{43}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{423}$ |
| $\mathbf{e}_{431}$ | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | 0 | $-\mathbf{e}_4$ | 0 | $\mathbf{e}_3$ | 0 | $-\mathbf{e}_1$ | $\mathbf{e}_{43}$ | 0 | $-\mathbf{e}_{41}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{431}$ |
| $\mathbf{e}_{412}$ | 0 | 0 | 0 | $-\mathbf{1}$ | 0 | 0 | 0 | $-\mathbf{e}_4$ | $-\mathbf{e}_2$ | $\mathbf{e}_1$ | 0 | $-\mathbf{e}_{42}$ | $\mathbf{e}_{41}$ | 0 | $\mathbf{e}_{12}$ | $\mathbf{e}_{412}$ |
| $\mathbf{e}_{321}$ | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | 0 | 0 | 0 | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | 0 | $\mathbf{e}_{321}$ |
| $\mathbb{1}$ | $\mathbf{1}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_4$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |

# Point

$$\mathbf{p} = \underbrace{p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3}_{\text{Position}} + \underbrace{p_w\mathbf{e}_4}_{\text{Weight}}$$



# Special Points

- The origin is simply the point $\mathbf{e}_4$

- Point with zero weight lies at infinity in ($x$, $y$, $z$) direction

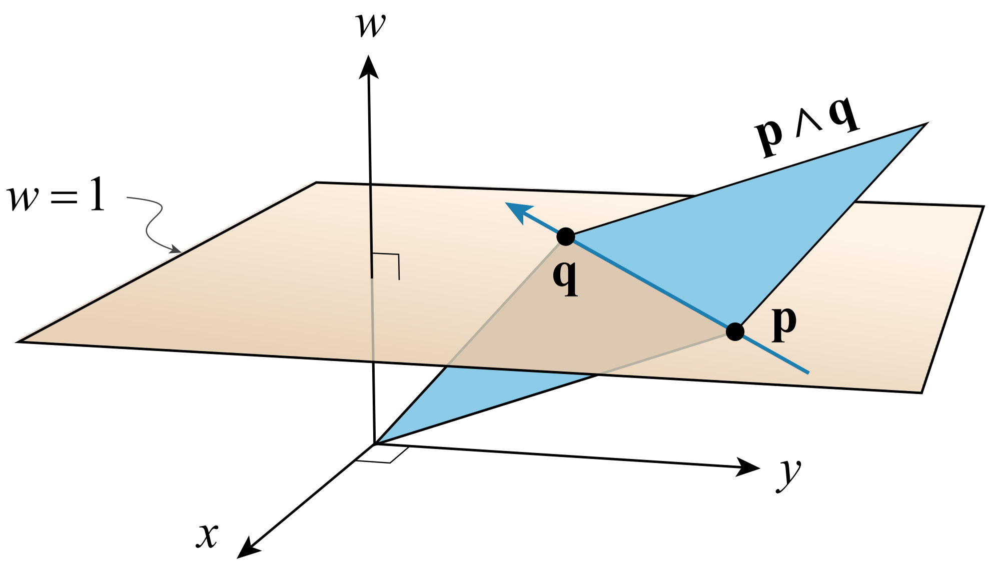
- Points at infinity in opposite directions are equivalent

# Line

$$\mathbf{p} \wedge \mathbf{q} = (q_x p_w - p_x q_w)\,\mathbf{e}_{41} + (q_y p_w - p_y q_w)\,\mathbf{e}_{42} + (q_z p_w - p_z q_w)\,\mathbf{e}_{43}$$
$$+ (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$$

$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}$$

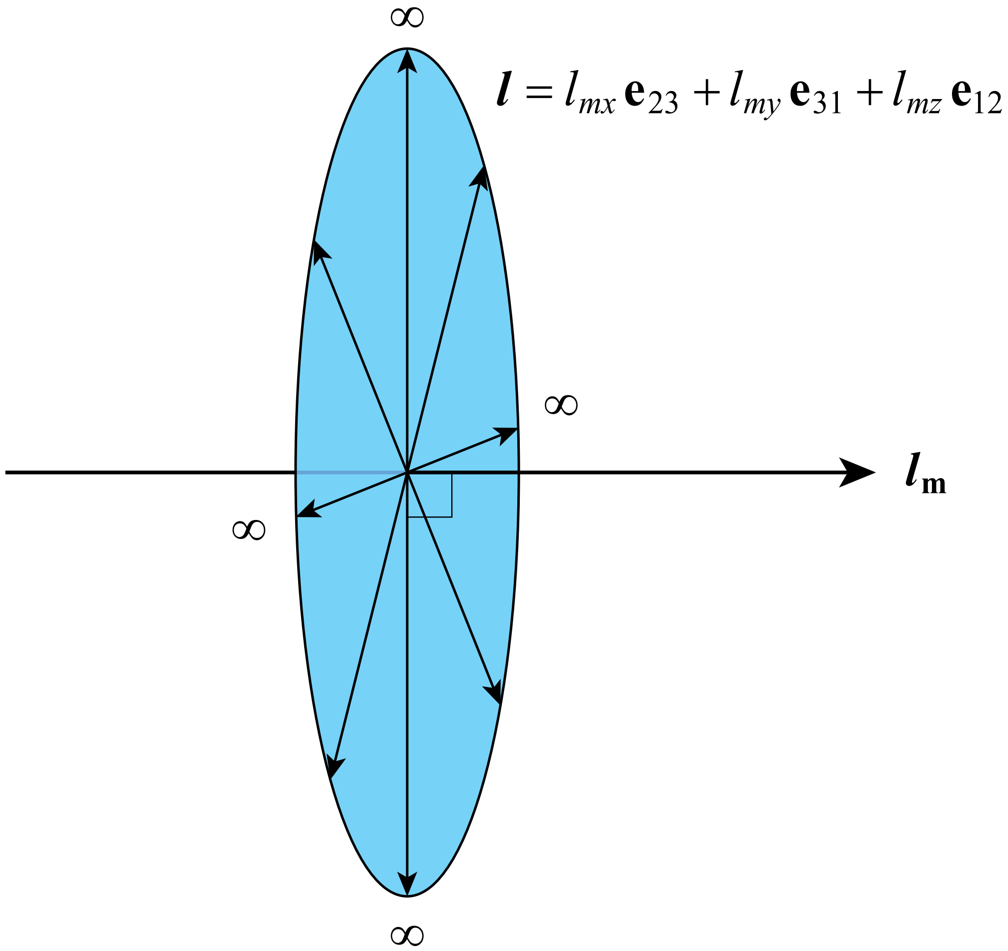
$$\boldsymbol{l}_{\mathbf{v}} \cdot \boldsymbol{l}_{\mathbf{m}} = 0$$



Sec. 2.4.2

# Lines at Infinity

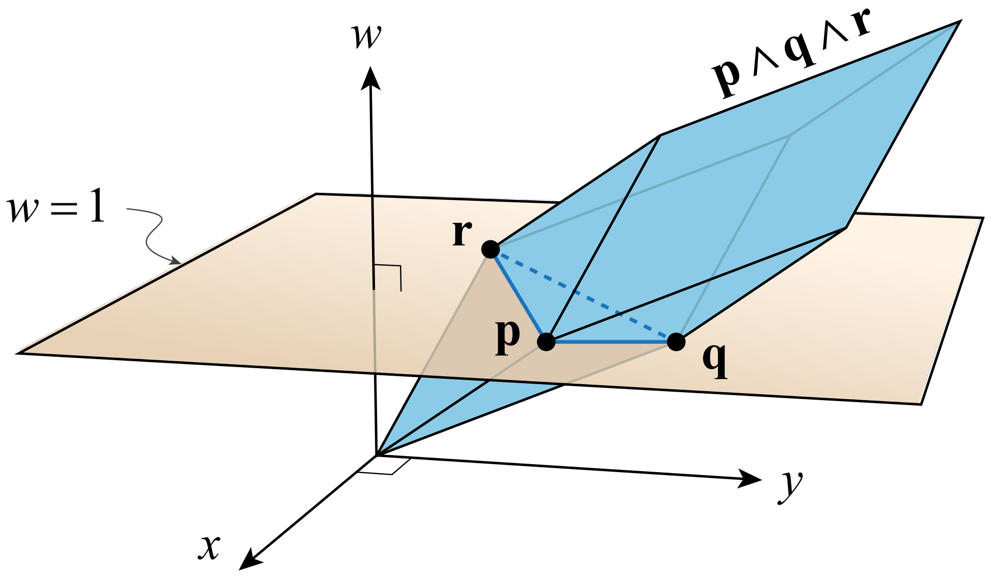
- Line with zero direction lies at infinity



# Plane

$$\boldsymbol{l} \wedge \mathbf{p} = (l_{vy}p_z - l_{vz}p_y + l_{mx})\,\overline{\mathbf{e}}_1 + (l_{vz}p_x - l_{vx}p_z + l_{my})\,\overline{\mathbf{e}}_2$$
$$+ (l_{vx}p_y - l_{vy}p_x + l_{mz})\,\overline{\mathbf{e}}_3 - (l_{mx}p_x + l_{my}p_y + l_{mz}p_z)\,\overline{\mathbf{e}}_4$$

$$\mathbf{g} = \underbrace{g_x\mathbf{e}_{423} + g_y\mathbf{e}_{431} + g_z\mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w\mathbf{e}_{321}}_{\text{Position}}$$



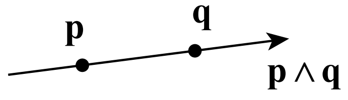
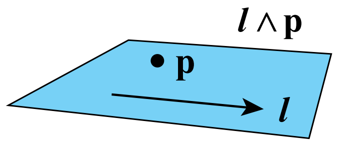
Sec. 2.4.3

# Horizon

- Plane with zero normal lies at infinity $g_w \mathbf{e}_{321}$
- Contains all points at infinity, all lines at infinity
- Given special name *horizon*
- Complement of origin

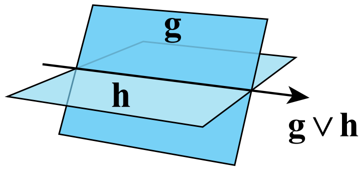
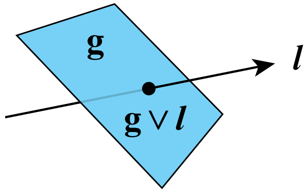
# Join

- Wedge product performs join operation

| **Join Operation** | **Illustration** |
|---|---|
| Line containing points **p** and **q**.<br>$\mathbf{p} \wedge \mathbf{q} = (p_w q_x - p_x q_w)\,\mathbf{e}_{41} + (p_w q_y - p_y q_w)\,\mathbf{e}_{42} + (p_w q_z - p_z q_w)\,\mathbf{e}_{43} + (p_y q_z - p_z q_y)\,\mathbf{e}_{23} + (p_z q_x - p_x q_z)\,\mathbf{e}_{31} + (p_x q_y - p_y q_x)\,\mathbf{e}_{12}$ |  |
| Plane containing line ***l*** and point **p**.<br>$\boldsymbol{l} \wedge \mathbf{p} = (l_{vy} p_z - l_{vz} p_y + l_{mx} p_w)\,\mathbf{e}_{423} + (l_{vz} p_x - l_{vx} p_z + l_{my} p_w)\,\mathbf{e}_{431} + (l_{vx} p_y - l_{vy} p_x + l_{mz} p_w)\,\mathbf{e}_{412} - (l_{mx} p_x + l_{my} p_y + l_{mz} p_z)\,\mathbf{e}_{321}$ |  |

# Meet

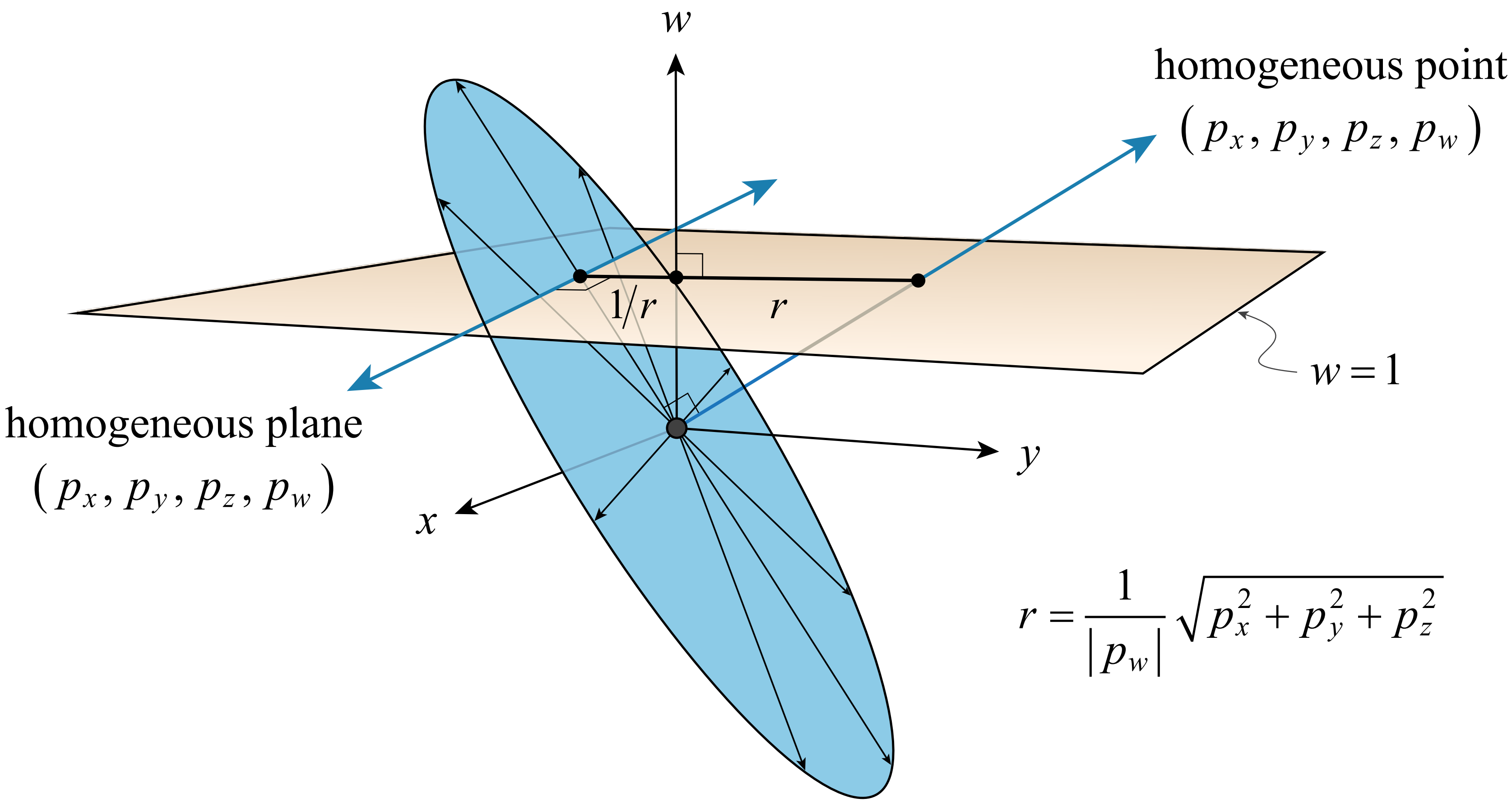
- Antiwedge product performs meet operation

| **Meet Operation** | **Illustration** |
|---|---|
| Line where planes **g** and **h** intersect.<br>$\mathbf{g} \vee \mathbf{h} = (g_z h_y - g_y h_z)\,\mathbf{e}_{41} + (g_x h_z - g_z h_x)\,\mathbf{e}_{42} + (g_y h_x - g_x h_y)\,\mathbf{e}_{43} + (g_x h_w - g_w h_x)\,\mathbf{e}_{23} + (g_y h_w - g_w h_y)\,\mathbf{e}_{31} + (g_z h_w - g_w h_z)\,\mathbf{e}_{12}$ |  |
| Point where plane **g** and line ***l*** intersect.<br>$\mathbf{g} \vee \boldsymbol{l} = (g_z l_{my} - g_y l_{mz} + g_w l_{vx})\,\mathbf{e}_1 + (g_x l_{mz} - g_z l_{mx} + g_w l_{vy})\,\mathbf{e}_2 + (g_y l_{mx} - g_x l_{my} + g_w l_{vz})\,\mathbf{e}_3 - (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})\,\mathbf{e}_4$ |  |

# Duality

- Every object can be interpreted as two different things
- Every operation performs two different actions
- One interpretation corresponds to regular space
- The other interpretation corresponds to *antispace*

# Duality



$$r = \frac{1}{|p_w|}\sqrt{p_x^2 + p_y^2 + p_z^2}$$

# Exomorphisms

- Given an *n* x *n* linear transformation **m** that operates on vectors
- The exomorphism **M** is the $2^n$ x $2^n$ matrix that operates on the whole algebra
- Exomorphism preserves structure under the wedge product:

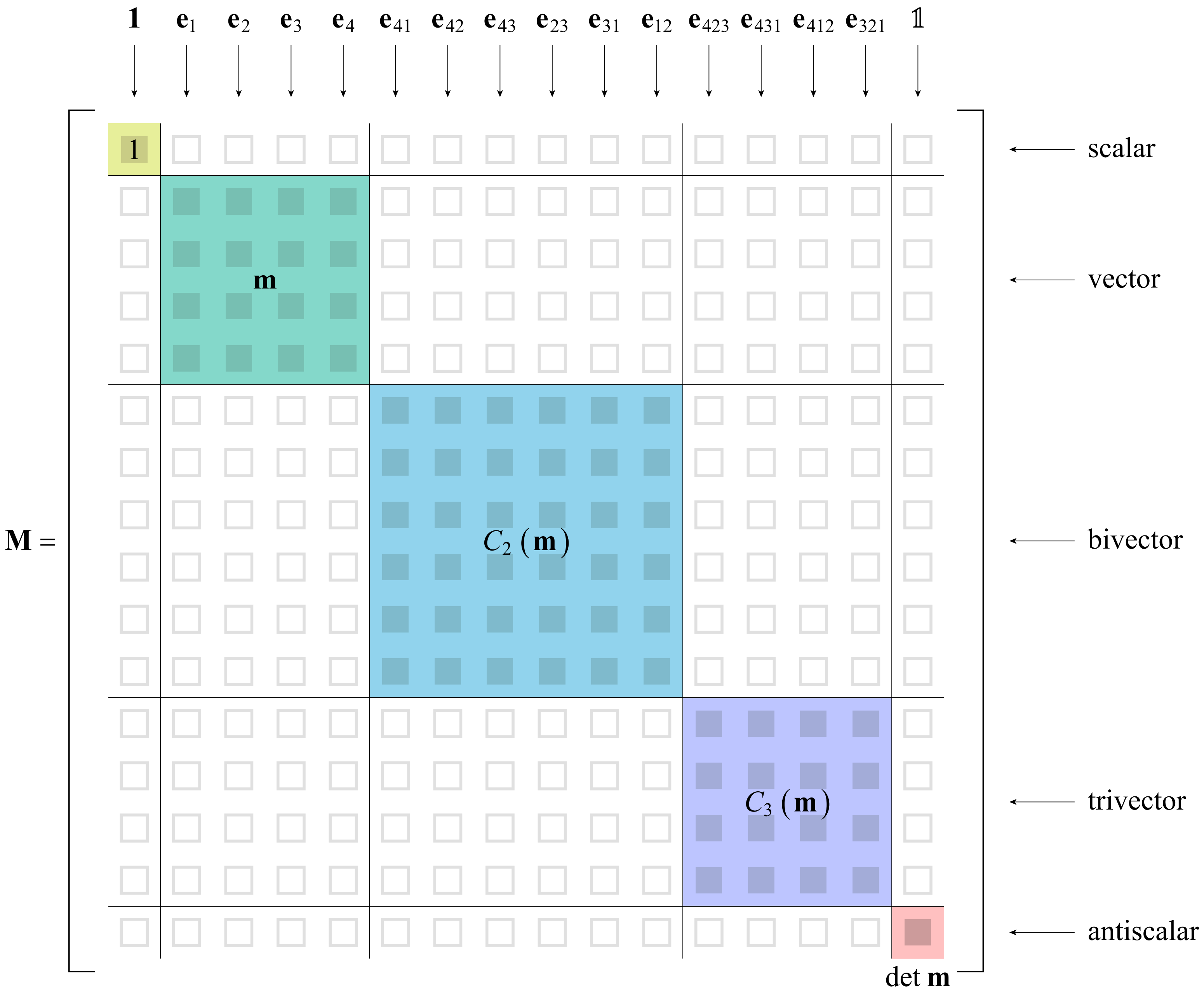
$$\mathbf{M}\left(\mathbf{a}\wedge\mathbf{b}\right)=\left(\mathbf{Ma}\right)\wedge\left(\mathbf{Mb}\right)$$

# Exomorphisms

- Matrix **M** is block diagonal

- Each block has columns given by wedge products of columns of the original matrix **m**

- These are called *compound matrices* of **m**


$\mathbf{M} =$
$\mathbf{1}$
$\mathbf{e}_1$
$\mathbf{e}_2$
$\mathbf{e}_3$
$\mathbf{e}_4$
$\mathbf{e}_{41}$
$\mathbf{e}_{42}$
$\mathbf{e}_{43}$
$\mathbf{e}_{23}$
$\mathbf{e}_{31}$
$\mathbf{e}_{12}$
$\mathbf{e}_{423}$
$\mathbf{e}_{431}$
$\mathbf{e}_{412}$
$\mathbf{e}_{321}$
$\mathbb{1}$
1
$\mathbf{m}$
$C_2(\mathbf{m})$
$C_3(\mathbf{m})$
$\det \mathbf{m}$
scalar
vector
bivector
trivector
antiscalar

# Translation Exomorphism

$$\mathbf{m} = \begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_2(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -t_z & t_y & 1 & 0 & 0 \\ t_z & 0 & -t_x & 0 & 1 & 0 \\ -t_y & t_x & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3(\mathbf{m}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -t_x & -t_y & -t_z & 1 \end{bmatrix}$$

# The Metric Tensor

- *n* x *n* matrix that defines dot products of vectors

$$\mathfrak{g} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = +1$$

$$\mathbf{e}_2 \cdot \mathbf{e}_2 = +1$$

$$\mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_4 \cdot \mathbf{e}_4 = 0$$

$$\mathfrak{g}_{ij} \equiv \mathbf{v}_i \cdot \mathbf{v}_j$$

# Metric Exomorphism

- The metric tensor is a linear transformation
- Thus, it can be extended to a full exomorphism matrix **G**
- There is also a metric *antiexomorphism*, or just “antimetric”, that satisfies

$$\mathbb{G}\mathbf{u} = \underline{\mathbf{G}\overline{\mathbf{u}}} = \overline{\mathbf{G}\underline{\mathbf{u}}}$$

Sec. 2.8.2

# Metric and Antimetric

$$\mathbf{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 1 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 1 & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0
\end{array}\right]$$

$$\mathbb{G} = \left[\begin{array}{c|cccc|cccccc|cccc|c}
0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\
\square & 0 & 0 & 0 & 1 & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\
\square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & \square & \square & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1 & 0 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 1 & 0 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 1 & 0 & \square \\
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 0 & 0 & 0 & 0 & \square \\ \hline
\square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & \square & 1
\end{array}\right]$$

$$\mathbf{G}\mathbb{G} = \det\left(\boldsymbol{\mathfrak{g}}\right)\mathbf{I}$$

# Bulk and Weight

- Multiplying $2^n$-dimensional multivector by metric or antimetric partitions into two pieces

- Bulk $\mathbf{u}_\bullet = \mathbf{G}\mathbf{u}$ All components without factor $\mathbf{e}_4$

- Weight $\mathbf{u}_\circ = \mathbb{G}\mathbf{u}$ All components with factor $\mathbf{e}_4$

Sec. 2.8.3

# Bulk and Weight of Point

$$\mathbf{p} = \underbrace{p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3}_{\text{Position}} + \underbrace{p_w \mathbf{e}_4}_{\text{Weight}}$$

$$\mathbf{p}_{\bullet} = p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3$$

$$\mathbf{p}_{\circ} = p_w \mathbf{e}_4$$

# Bulk and Weight of Line

$$\boldsymbol{l} = \underbrace{l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}}_{\text{Direction}} + \underbrace{l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}}_{\text{Moment}}$$

$$\boldsymbol{l}_{\bullet} = l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$$

$$\boldsymbol{l}_{\circ} = l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43}$$

# Bulk and Weight of Plane

$$\mathbf{g} = \underbrace{g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}}_{\text{Normal}} + \underbrace{g_w \mathbf{e}_{321}}_{\text{Position}}$$

$$\mathbf{g}_{\bullet} = g_w \mathbf{e}_{321}$$

$$\mathbf{g}_{\circ} = g_x \mathbf{e}_{423} + g_y \mathbf{e}_{431} + g_z \mathbf{e}_{412}$$

# Bulk and Weight

- Bulk contains positional information

- Weight contains directional information

- If the bulk is zero, then the object contains the origin

- If the weight zero, then the horizon contains the object

# Inner Product

- Dot product defined by metric:

$$\mathbf{a} \bullet \mathbf{b} = \left( \mathbf{a}^{\mathrm{T}} \mathbf{G} \mathbf{b} \right) \mathbf{1}$$

- Antidot product defined by antimetric:

$$\mathbf{a} \circ \mathbf{b} = \left( \mathbf{a}^{\mathrm{T}} \mathbb{G} \mathbf{b} \right) \mathbb{1}$$

- Satisfies De Morgan law:

$$\mathbf{a} \circ \mathbf{b} = \overline{\underline{\mathbf{a}} \bullet \underline{\mathbf{b}}}$$

Sec. 2.9

# Bulk and Weight Norms

- Two dot products produce two norms
- Bulk norm: $\left\|\mathbf{u}\right\|_{\bullet} = \sqrt{\mathbf{u} \bullet \mathbf{u}}$
- Weight norm: $\left\|\mathbf{u}\right\|_{\circ} = \sqrt{\mathbf{u} \circ \mathbf{u}}$

# Bulk and Weight Norms

| **Type** | **Bulk Norm** | **Weight Norm** |
|---|---|---|
| Point **p** | $\|\mathbf{p}\|_\bullet = \mathbf{1}\sqrt{p_x^2 + p_y^2 + p_z^2}$ | $\|\mathbf{p}\|_\circ = \lvert p_w \rvert \mathbb{1}$ |
| Line ***l*** | $\|\boldsymbol{l}\|_\bullet = \mathbf{1}\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}$ | $\|\boldsymbol{l}\|_\circ = \mathbb{1}\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}$ |
| Plane **g** | $\|\mathbf{g}\|_\bullet = \lvert g_w \rvert \mathbf{1}$ | $\|\mathbf{g}\|_\circ = \mathbb{1}\sqrt{g_x^2 + g_y^2 + g_z^2}$ |

# Unitization

- An object is *unitized* when its weight has magnitude one

| **Type** | **Definition** | **Unitization** |
|---|---|---|
| Point $\mathbf{p}$ | $\mathbf{p} = p_x\,\mathbf{e}_1 + p_y\,\mathbf{e}_2 + p_z\,\mathbf{e}_3 + p_w\,\mathbf{e}_4$ | $p_w^2 = 1$ |
| Line $\boldsymbol{l}$ | $\boldsymbol{l} = l_{vx}\,\mathbf{e}_{41} + l_{vy}\,\mathbf{e}_{42} + l_{vz}\,\mathbf{e}_{43} + l_{mx}\,\mathbf{e}_{23} + l_{my}\,\mathbf{e}_{31} + l_{mz}\,\mathbf{e}_{12}$ | $l_{vx}^2 + l_{vy}^2 + l_{vz}^2 = 1$ |
| Plane $\mathbf{g}$ | $\mathbf{g} = g_x\,\mathbf{e}_{423} + g_y\,\mathbf{e}_{431} + g_z\,\mathbf{e}_{412} + g_w\,\mathbf{e}_{321}$ | $g_x^2 + g_y^2 + g_z^2 = 1$ |

# Geometric Norm

- Bulk and weight norms by themselves not meaningful
- But add them, and result is a *homogeneous magnitude*
- Represents distance from origin
- Called the geometric norm

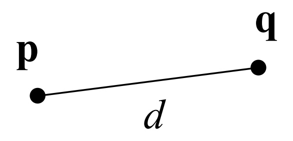
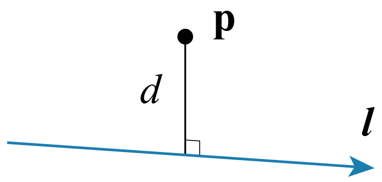
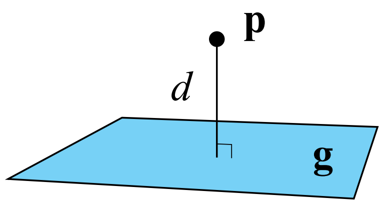
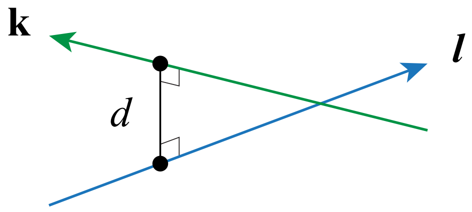
$$\|\mathbf{u}\| = \|\mathbf{u}\|_{\bullet} + \|\mathbf{u}\|_{\circ} = \sqrt{\mathbf{u} \bullet \mathbf{u}} + \sqrt{\mathbf{u} \circ \mathbf{u}}$$

- Can be unitized by making weight one

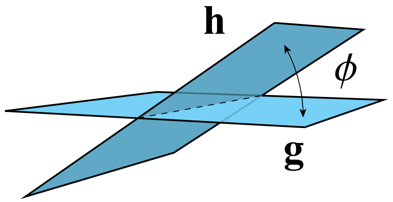
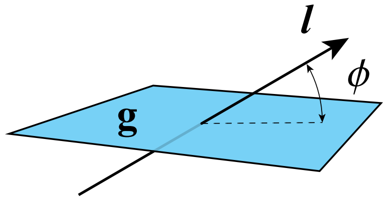
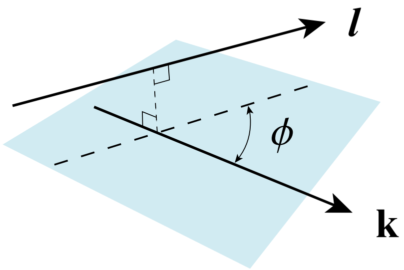
# Geometric Norm

| **Type** | **Geometric Norm** | **Interpretation** |
|---|---|---|
| Point **p** | $\widehat{\lVert \mathbf{p} \rVert} = \dfrac{\sqrt{p_x^2 + p_y^2 + p_z^2}}{\lvert p_w \rvert}$ | Distance from the origin to the point **p**. |
| Line ***l*** | $\widehat{\lVert \boldsymbol{l} \rVert} = \dfrac{\sqrt{l_{mx}^2 + l_{my}^2 + l_{mz}^2}}{\sqrt{l_{vx}^2 + l_{vy}^2 + l_{vz}^2}}$ | Perpendicular distance from the origin to the line ***l***. |
| Plane **g** | $\widehat{\lVert \mathbf{g} \rVert} = \dfrac{\lvert g_w \rvert}{\sqrt{g_x^2 + g_y^2 + g_z^2}}$ | Perpendicular distance from the origin to the plane **g**. |

# Euclidean Distance

| **Distance Formula** | **Illustration** |
|---|---|
| Distance $d$ between points **p** and **q**.<br>$d(\mathbf{p},\mathbf{q}) = \lVert \mathbf{q}_{xyz} p_w - \mathbf{p}_{xyz} q_w \rVert \mathbf{1} + \lvert p_w q_w \rvert \mathbb{1}$ |  |
| Perpendicular distance $d$ between point **p** and line ***l***.<br>$d(\mathbf{p},\boldsymbol{l}) = \lVert \boldsymbol{l}_{\mathbf{v}} \times \mathbf{p}_{xyz} + p_w \boldsymbol{l}_{\mathbf{m}} \rVert \mathbf{1} + \lVert p_w \boldsymbol{l}_{\mathbf{v}} \rVert \mathbb{1}$ |  |
| Perpendicular distance $d$ between point **p** and plane **g**.<br>$d(\mathbf{p},\mathbf{g}) = (\mathbf{p} \cdot \mathbf{g}) \mathbf{1} + \lVert p_w \mathbf{g}_{xyz} \rVert \mathbb{1}$ |  |
| Perpendicular distance $d$ between skew lines ***l*** and **k**.<br>$d(\boldsymbol{l},\mathbf{k}) = -(\boldsymbol{l}_{\mathbf{v}} \cdot \mathbf{k}_{\mathbf{m}} + \boldsymbol{l}_{\mathbf{m}} \cdot \mathbf{k}_{\mathbf{v}}) \mathbf{1} + \lVert \boldsymbol{l}_{\mathbf{v}} \times \mathbf{k}_{\mathbf{v}} \rVert \mathbb{1}$ |  |

# Euclidean Angle

| **Angle Formula** | **Illustration** |
|---|---|
| Cosine of angle $\phi$ between planes $\mathbf{g}$ and $\mathbf{h}$.<br>$\cos\phi(\mathbf{g},\mathbf{h}) = (\mathbf{g}_{xyz} \cdot \mathbf{h}_{xyz})\mathbf{1} + \lVert\mathbf{g}\rVert_{\circ}\lVert\mathbf{h}\rVert_{\circ}$ |  |
| Cosine of angle $\phi$ between plane $\mathbf{g}$ and line $\boldsymbol{l}$.<br>$\cos\phi(\mathbf{g},\boldsymbol{l}) = \lVert\mathbf{g}_{xyz} \times \boldsymbol{l}_{\mathrm{v}}\rVert\mathbf{1} + \lVert\mathbf{g}\rVert_{\circ}\lVert\boldsymbol{l}\rVert_{\circ}$ |  |
| Cosine of angle $\phi$ between lines $\boldsymbol{l}$ and $\mathbf{k}$.<br>$\cos\phi(\boldsymbol{l},\mathbf{k}) = (\boldsymbol{l}_{\mathrm{v}} \cdot \mathbf{k}_{\mathrm{v}})\mathbf{1} + \lVert\boldsymbol{l}\rVert_{\circ}\lVert\mathbf{k}\rVert_{\circ}$ |  |

# Bulk and Weight Duals

- Multiply by metric or antimetric, then take complement
- Bulk dual: $\mathbf{u}^{\bigstar} = \overline{\mathbf{G}\mathbf{u}}$
- Weight dual: $\mathbf{u}^{\star} = \overline{\mathbb{G}\mathbf{u}}$

# Bulk and Weight Duals

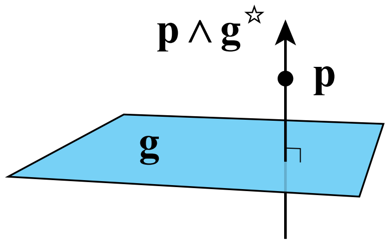
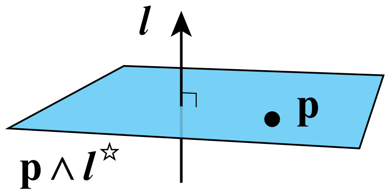
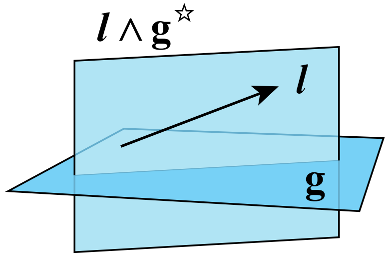
| $\mathbf{u}$ | $\mathbf{1}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_4$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{u}^{\bigstar}$ | $\mathbb{1}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $0$ | $0$ | $0$ | $0$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $0$ | $0$ | $0$ | $-\mathbf{e}_4$ | $0$ |
| $\mathbf{u}_{\bigstar}$ | $\mathbb{1}$ | $-\mathbf{e}_{423}$ | $-\mathbf{e}_{431}$ | $-\mathbf{e}_{412}$ | $0$ | $0$ | $0$ | $0$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $0$ | $0$ | $0$ | $\mathbf{e}_4$ | $0$ |
| $\mathbf{u}^{\text{☆}}$ | $0$ | $0$ | $0$ | $0$ | $\mathbf{e}_{321}$ | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | $0$ | $0$ | $0$ | $-\mathbf{e}_1$ | $-\mathbf{e}_2$ | $-\mathbf{e}_3$ | $0$ | $\mathbf{1}$ |
| $\mathbf{u}_{\text{☆}}$ | $0$ | $0$ | $0$ | $0$ | $-\mathbf{e}_{321}$ | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | $0$ | $0$ | $0$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $0$ | $\mathbf{1}$ |

# Interior Products

- Two exterior products combined with two duals
- Four *interior* products

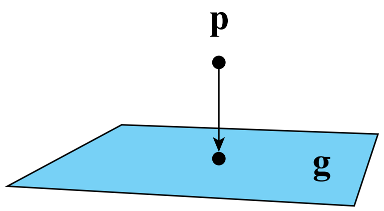
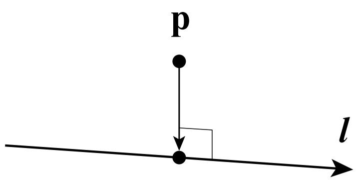
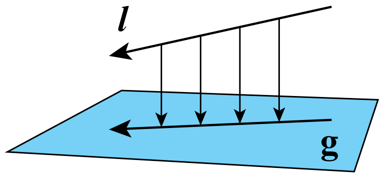
- Bulk contraction $\mathbf{a} \vee \mathbf{b}^{\bigstar}$
- Weight contraction $\mathbf{a} \vee \mathbf{b}^{\text{☆}}$
- Bulk expansion $\mathbf{a} \wedge \mathbf{b}^{\bigstar}$
- Weight expansion $\mathbf{a} \wedge \mathbf{b}^{\text{☆}}$

# Weight Expansion

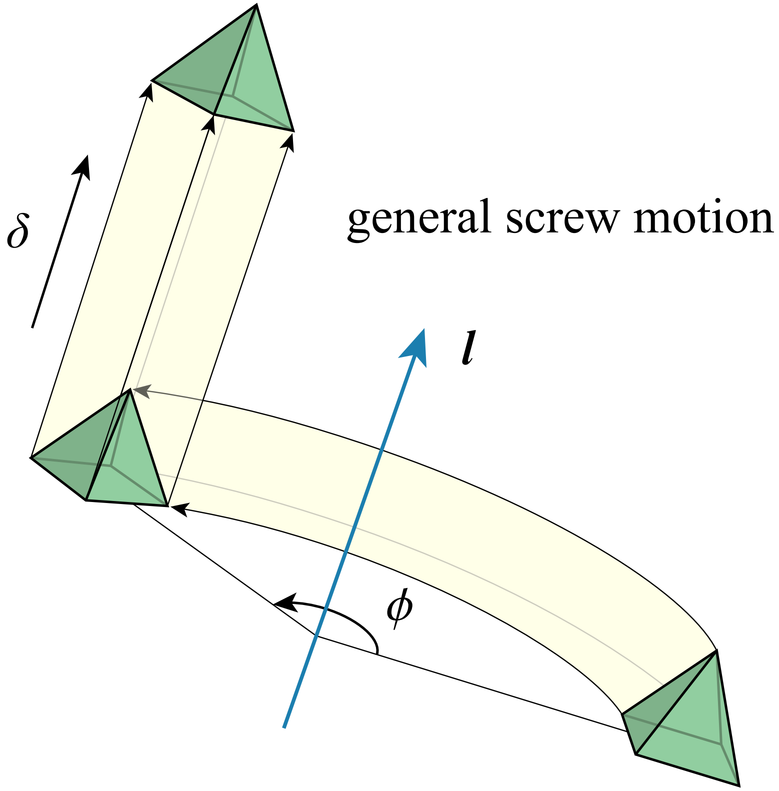
| **Expansion Operation** | **Illustration** |
|---|---|
| Line containing point **p** and orthogonal to plane **g**.<br>$\mathbf{p} \wedge \mathbf{g}^{\star} = -p_w g_x \mathbf{e}_{41} - p_w g_y \mathbf{e}_{42} - p_w g_z \mathbf{e}_{43}$<br>$+ (p_z g_y - p_y g_z)\,\mathbf{e}_{23} + (p_x g_z - p_z g_x)\,\mathbf{e}_{31} + (p_y g_x - p_x g_y)\,\mathbf{e}_{12}$ |  |
| Plane containing point **p** and orthogonal to line ***l***.<br>$\mathbf{p} \wedge \boldsymbol{l}^{\star} = -p_w l_{vx} \mathbf{e}_{423} - p_w l_{vy} \mathbf{e}_{431} - p_w l_{vz} \mathbf{e}_{412}$<br>$+ (p_x l_{vx} + p_y l_{vy} + p_z l_{vz})\,\mathbf{e}_{321}$ |  |
| Plane containing line ***l*** and orthogonal to plane **g**.<br>$\boldsymbol{l} \wedge \mathbf{g}^{\star} = (l_{vy} g_z - l_{vz} g_y)\,\mathbf{e}_{423} + (l_{vz} g_x - l_{vx} g_z)\,\mathbf{e}_{431} + (l_{vx} g_y - l_{vy} g_x)\,\mathbf{e}_{412}$<br>$- (l_{mx} g_x + l_{my} g_y + l_{mz} g_z)\,\mathbf{e}_{321}$ |  |

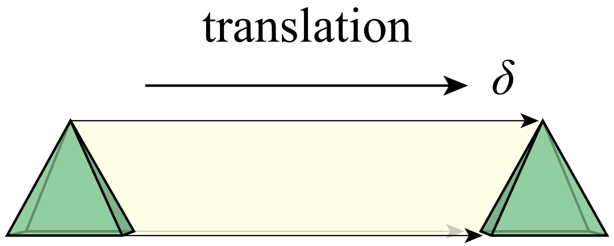
Sec. 2.13.5

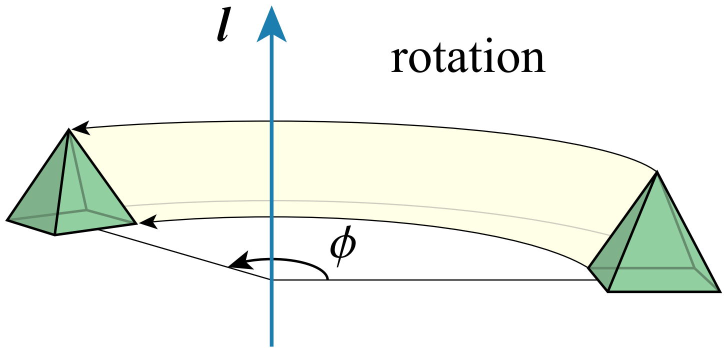
# Orthogonal Projection

| **Projection Operation** | **Illustration** |
|---|---|
| Orthogonal projection of point **p** onto plane **g**.<br>$\mathbf{g} \vee (\mathbf{p} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(p_x \mathbf{e}_1 + p_y \mathbf{e}_2 + p_z \mathbf{e}_3 + p_w \mathbf{e}_4)$<br>$- (g_x p_x + g_y p_y + g_z p_z + g_w p_w)(g_x \mathbf{e}_1 + g_y \mathbf{e}_2 + g_z \mathbf{e}_3)$ |  |
| Orthogonal projection of point **p** onto line ***l***.<br>$\boldsymbol{l} \vee (\mathbf{p} \wedge \boldsymbol{l}^{\star}) = (l_{vx} p_x + l_{vy} p_y + l_{vz} p_z)(l_{vx} \mathbf{e}_1 + l_{vy} \mathbf{e}_2 + l_{vz} \mathbf{e}_3) + (l_{vx}^2 + l_{vy}^2 + l_{vz}^2)\, p_w \mathbf{e}_4$<br>$+ (l_{vy} l_{mz} - l_{vz} l_{my})\, p_w \mathbf{e}_1 + (l_{vz} l_{mx} - l_{vx} l_{mz})\, p_w \mathbf{e}_2 + (l_{vx} l_{my} - l_{vy} l_{mx})\, p_w \mathbf{e}_3$ |  |
| Orthogonal projection of line ***l*** onto plane **g**.<br>$\mathbf{g} \vee (\boldsymbol{l} \wedge \mathbf{g}^{\star}) = (g_x^2 + g_y^2 + g_z^2)(l_{vx} \mathbf{e}_{41} + l_{vy} \mathbf{e}_{42} + l_{vz} \mathbf{e}_{43})$<br>$- (g_x l_{vx} + g_y l_{vy} + g_z l_{vz})(g_x \mathbf{e}_{41} + g_y \mathbf{e}_{42} + g_z \mathbf{e}_{43})$<br>$+ (g_x l_{mx} + g_y l_{my} + g_z l_{mz})(g_x \mathbf{e}_{23} + g_y \mathbf{e}_{31} + g_z \mathbf{e}_{12})$<br>$+ (g_z l_{vy} - g_y l_{vz})\, g_w \mathbf{e}_{23} + (g_x l_{vz} - g_z l_{vx})\, g_w \mathbf{e}_{31} + (g_y l_{vx} - g_x l_{vy})\, g_w \mathbf{e}_{12}$ |  |

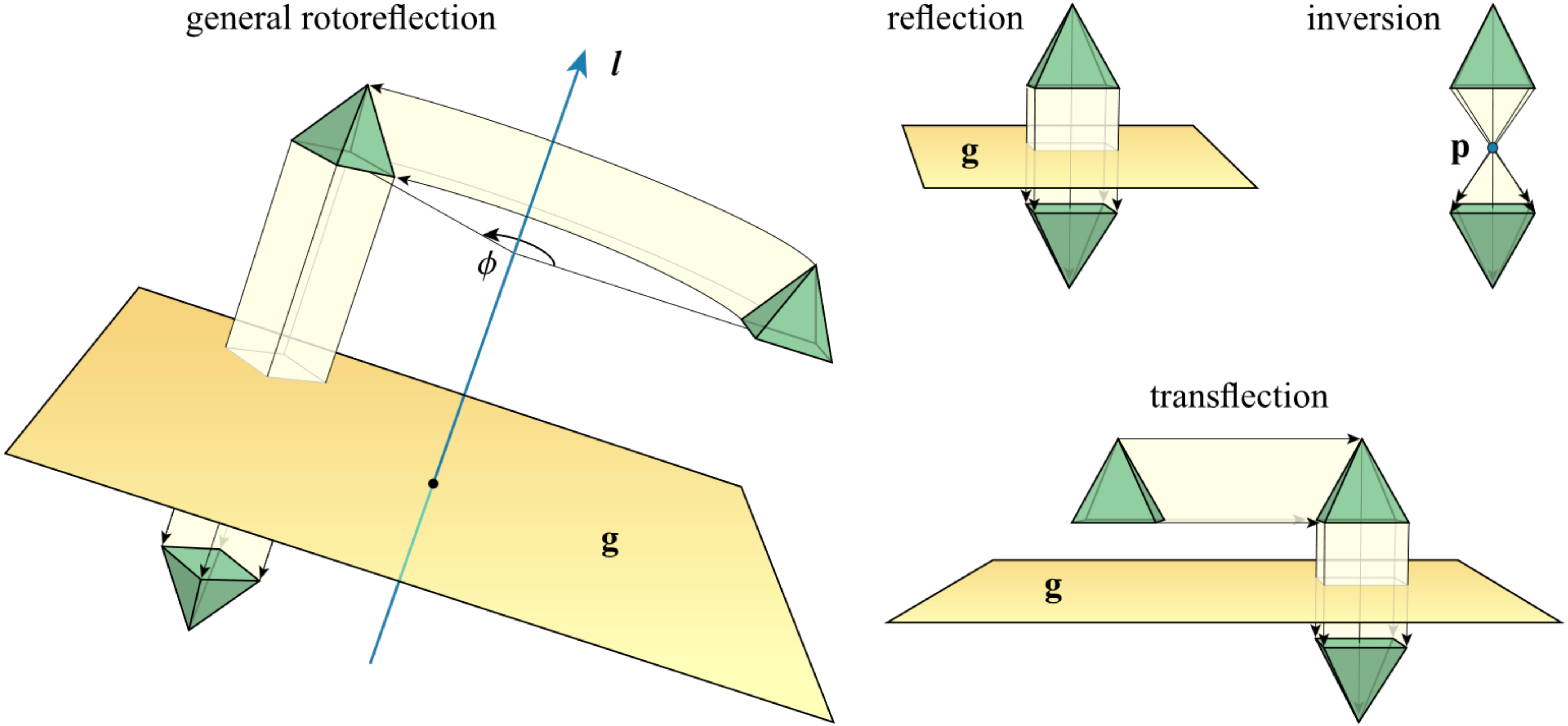
# Proper Euclidean Isometries







# Improper Euclidean Isometries



# Geometric Product

- Historically denoted by juxtaposition without symbol
- But there is always product and antiproduct
- We use upward and downward wedge with dot inside

- Geometric product $\mathbf{a} \mathbin{⟑} \mathbf{b}$

- Geometric antiproduct $\mathbf{a} \mathbin{⟇} \mathbf{b}$

- “Wedge-dot” and “Antiwedge-dot”

# Geometric Product

- Defined by slightly different property compared to exterior product
- For vectors, $\mathbf{v} \mathbin{⟑} \mathbf{v} = \mathbf{v} \bullet \mathbf{v}$
- Geometric product depends on the metric
- **1** is the identity element

# 4D Geometric Product

Geometric Product $\mathbf{a} ⟑ \mathbf{b}$

| $\mathbf{b}$<br>$\mathbf{a}$ | $\mathbf{1}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{1}$ | $\mathbf{1}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
| $\mathbf{e}_{1}$ | $\mathbf{e}_{1}$ | $\mathbf{1}$ | $\mathbf{e}_{12}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{412}$ | $\mathbf{e}_{431}$ | $-\mathbf{e}_{321}$ | $-\mathbf{e}_{3}$ | $\mathbf{e}_{2}$ | $\mathbb{1}$ | $\mathbf{e}_{43}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{23}$ | $\mathbf{e}_{423}$ |
| $\mathbf{e}_{2}$ | $\mathbf{e}_{2}$ | $-\mathbf{e}_{12}$ | $\mathbf{1}$ | $\mathbf{e}_{23}$ | $-\mathbf{e}_{42}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{423}$ | $\mathbf{e}_{3}$ | $-\mathbf{e}_{321}$ | $-\mathbf{e}_{1}$ | $-\mathbf{e}_{43}$ | $\mathbb{1}$ | $\mathbf{e}_{41}$ | $-\mathbf{e}_{31}$ | $\mathbf{e}_{431}$ |
| $\mathbf{e}_{3}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{31}$ | $-\mathbf{e}_{23}$ | $\mathbf{1}$ | $-\mathbf{e}_{43}$ | $-\mathbf{e}_{431}$ | $\mathbf{e}_{423}$ | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{2}$ | $\mathbf{e}_{1}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{42}$ | $-\mathbf{e}_{41}$ | $\mathbb{1}$ | $-\mathbf{e}_{12}$ | $\mathbf{e}_{412}$ |
| $\mathbf{e}_{4}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | 0 | 0 | 0 | $\mathbb{1}$ | 0 |
| $\mathbf{e}_{41}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{431}$ | 0 | 0 | 0 | 0 | $-\mathbb{1}$ | $-\mathbf{e}_{43}$ | $\mathbf{e}_{42}$ | 0 | 0 | 0 | $-\mathbf{e}_{423}$ | 0 |
| $\mathbf{e}_{42}$ | $\mathbf{e}_{42}$ | $-\mathbf{e}_{412}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{423}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{43}$ | $-\mathbb{1}$ | $-\mathbf{e}_{41}$ | 0 | 0 | 0 | $-\mathbf{e}_{431}$ | 0 |
| $\mathbf{e}_{43}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{431}$ | $-\mathbf{e}_{423}$ | $\mathbf{e}_{4}$ | 0 | 0 | 0 | 0 | $-\mathbf{e}_{42}$ | $\mathbf{e}_{41}$ | $-\mathbb{1}$ | 0 | 0 | 0 | $-\mathbf{e}_{412}$ | 0 |
| $\mathbf{e}_{23}$ | $\mathbf{e}_{23}$ | $-\mathbf{e}_{321}$ | $-\mathbf{e}_{3}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{423}$ | $-\mathbb{1}$ | $-\mathbf{e}_{43}$ | $\mathbf{e}_{42}$ | $-\mathbf{1}$ | $-\mathbf{e}_{12}$ | $\mathbf{e}_{31}$ | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{412}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{41}$ |
| $\mathbf{e}_{31}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{3}$ | $-\mathbf{e}_{321}$ | $-\mathbf{e}_{1}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{43}$ | $-\mathbb{1}$ | $-\mathbf{e}_{41}$ | $\mathbf{e}_{12}$ | $-\mathbf{1}$ | $-\mathbf{e}_{23}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{423}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{42}$ |
| $\mathbf{e}_{12}$ | $\mathbf{e}_{12}$ | $-\mathbf{e}_{2}$ | $\mathbf{e}_{1}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{42}$ | $\mathbf{e}_{41}$ | $-\mathbb{1}$ | $-\mathbf{e}_{31}$ | $\mathbf{e}_{23}$ | $-\mathbf{1}$ | $-\mathbf{e}_{431}$ | $\mathbf{e}_{423}$ | $-\mathbf{e}_{4}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{43}$ |
| $\mathbf{e}_{423}$ | $\mathbf{e}_{423}$ | $-\mathbb{1}$ | $-\mathbf{e}_{43}$ | $\mathbf{e}_{42}$ | 0 | 0 | 0 | 0 | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{412}$ | $\mathbf{e}_{431}$ | 0 | 0 | 0 | $\mathbf{e}_{41}$ | 0 |
| $\mathbf{e}_{431}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{43}$ | $-\mathbb{1}$ | $-\mathbf{e}_{41}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{412}$ | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{423}$ | 0 | 0 | 0 | $\mathbf{e}_{42}$ | 0 |
| $\mathbf{e}_{412}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{42}$ | $\mathbf{e}_{41}$ | $-\mathbb{1}$ | 0 | 0 | 0 | 0 | $-\mathbf{e}_{431}$ | $\mathbf{e}_{423}$ | $-\mathbf{e}_{4}$ | 0 | 0 | 0 | $\mathbf{e}_{43}$ | 0 |
| $\mathbf{e}_{321}$ | $\mathbf{e}_{321}$ | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | $-\mathbb{1}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $-\mathbf{1}$ | $\mathbf{e}_{4}$ |
| $\mathbb{1}$ | $\mathbb{1}$ | $-\mathbf{e}_{423}$ | $-\mathbf{e}_{431}$ | $-\mathbf{e}_{412}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | 0 | 0 | 0 | $-\mathbf{e}_{4}$ | 0 |

# Geometric Antiproduct

- Defined by De Morgan law:

$$\mathbf{a} ⟇ \mathbf{b} = \overline{\underline{\mathbf{a}} ⟑ \underline{\mathbf{b}}}$$

- Antivector **u** squares to antidot product:

$$\mathbf{u} ⟇ \mathbf{u} = \mathbf{u} \circ \mathbf{u}$$

- $\mathbb{1}$ is the identity element

# 4D Geometric Antiproduct

Geometric Antiproduct $\mathbf{a}$ ⟇ $\mathbf{b}$

| $\mathbf{a}$ \ $\mathbf{b}$ | $\mathbf{1}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mathbf{1}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{321}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | 0 | 0 | 0 | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | 0 | $\mathbf{1}$ |
| $\mathbf{e}_{1}$ | 0 | 0 | 0 | 0 | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{3}$ | $-\mathbf{e}_{2}$ | 0 | 0 | 0 | $\mathbf{1}$ | $-\mathbf{e}_{12}$ | $\mathbf{e}_{31}$ | 0 | $\mathbf{e}_{1}$ |
| $\mathbf{e}_{2}$ | 0 | 0 | 0 | 0 | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{3}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{1}$ | 0 | 0 | 0 | $\mathbf{e}_{12}$ | $\mathbf{1}$ | $-\mathbf{e}_{23}$ | 0 | $\mathbf{e}_{2}$ |
| $\mathbf{e}_{3}$ | 0 | 0 | 0 | 0 | $-\mathbf{e}_{12}$ | $\mathbf{e}_{2}$ | $-\mathbf{e}_{1}$ | $-\mathbf{e}_{321}$ | 0 | 0 | 0 | $-\mathbf{e}_{31}$ | $\mathbf{e}_{23}$ | $\mathbf{1}$ | 0 | $\mathbf{e}_{3}$ |
| $\mathbf{e}_{4}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $-\mathbb{1}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{1}$ | $-\mathbf{e}_{2}$ | $-\mathbf{e}_{3}$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $\mathbf{1}$ | $\mathbf{e}_{4}$ |
| $\mathbf{e}_{41}$ | $\mathbf{e}_{23}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{3}$ | $-\mathbf{e}_{2}$ | $\mathbf{e}_{423}$ | $-\mathbb{1}$ | $\mathbf{e}_{43}$ | $-\mathbf{e}_{42}$ | $-\mathbf{1}$ | $\mathbf{e}_{12}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{4}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{431}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{41}$ |
| $\mathbf{e}_{42}$ | $\mathbf{e}_{31}$ | $-\mathbf{e}_{3}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{431}$ | $-\mathbf{e}_{43}$ | $-\mathbb{1}$ | $\mathbf{e}_{41}$ | $-\mathbf{e}_{12}$ | $-\mathbf{1}$ | $\mathbf{e}_{23}$ | $-\mathbf{e}_{412}$ | $-\mathbf{e}_{4}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{42}$ |
| $\mathbf{e}_{43}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{2}$ | $-\mathbf{e}_{1}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{42}$ | $-\mathbf{e}_{41}$ | $-\mathbb{1}$ | $\mathbf{e}_{31}$ | $-\mathbf{e}_{23}$ | $-\mathbf{1}$ | $\mathbf{e}_{431}$ | $-\mathbf{e}_{423}$ | $-\mathbf{e}_{4}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{43}$ |
| $\mathbf{e}_{23}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{1}$ | $-\mathbf{1}$ | $\mathbf{e}_{12}$ | $-\mathbf{e}_{31}$ | 0 | 0 | 0 | $-\mathbf{e}_{321}$ | $\mathbf{e}_{3}$ | $-\mathbf{e}_{2}$ | 0 | $\mathbf{e}_{23}$ |
| $\mathbf{e}_{31}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{2}$ | $-\mathbf{e}_{12}$ | $-\mathbf{1}$ | $\mathbf{e}_{23}$ | 0 | 0 | 0 | $-\mathbf{e}_{3}$ | $-\mathbf{e}_{321}$ | $\mathbf{e}_{1}$ | 0 | $\mathbf{e}_{31}$ |
| $\mathbf{e}_{12}$ | 0 | 0 | 0 | 0 | $\mathbf{e}_{3}$ | $\mathbf{e}_{31}$ | $-\mathbf{e}_{23}$ | $-\mathbf{1}$ | 0 | 0 | 0 | $\mathbf{e}_{2}$ | $-\mathbf{e}_{1}$ | $-\mathbf{e}_{321}$ | 0 | $\mathbf{e}_{12}$ |
| $\mathbf{e}_{423}$ | $-\mathbf{e}_{1}$ | $-\mathbf{1}$ | $\mathbf{e}_{12}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{4}$ | $\mathbf{e}_{412}$ | $-\mathbf{e}_{431}$ | $\mathbf{e}_{321}$ | $-\mathbf{e}_{3}$ | $\mathbf{e}_{2}$ | $\mathbb{1}$ | $-\mathbf{e}_{43}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{423}$ |
| $\mathbf{e}_{431}$ | $-\mathbf{e}_{2}$ | $-\mathbf{e}_{12}$ | $-\mathbf{1}$ | $\mathbf{e}_{23}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{412}$ | $-\mathbf{e}_{4}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{321}$ | $-\mathbf{e}_{1}$ | $\mathbf{e}_{43}$ | $\mathbb{1}$ | $-\mathbf{e}_{41}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{431}$ |
| $\mathbf{e}_{412}$ | $-\mathbf{e}_{3}$ | $\mathbf{e}_{31}$ | $-\mathbf{e}_{23}$ | $-\mathbf{1}$ | $-\mathbf{e}_{43}$ | $\mathbf{e}_{431}$ | $-\mathbf{e}_{423}$ | $-\mathbf{e}_{4}$ | $-\mathbf{e}_{2}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{321}$ | $-\mathbf{e}_{42}$ | $\mathbf{e}_{41}$ | $\mathbb{1}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{412}$ |
| $\mathbf{e}_{321}$ | 0 | 0 | 0 | 0 | $-\mathbf{1}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | 0 | 0 | 0 | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | 0 | $\mathbf{e}_{321}$ |
| $\mathbb{1}$ | $\mathbf{1}$ | $\mathbf{e}_{1}$ | $\mathbf{e}_{2}$ | $\mathbf{e}_{3}$ | $\mathbf{e}_{4}$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |

# Geometric Product

- Geometric **product** in 4D space fixes the origin
- Cannot perform transformations we want

- Geometric **antiproduct** performs Euclidean isometries
- Uses sandwiching similar to quaternions

# Plane Reflection

- Sandwich antiproduct with plane **g** performs reflection:

$$\mathbf{u}' = \mathbf{g} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \mathbf{g}$$

- Multiple reflections stack outward from **u**:

$$\mathbf{u}' = \left(\mathbf{h} \mathbin{\dot{\vee}} \mathbf{g}\right) \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \left(\mathbf{g} \mathbin{\dot{\vee}} \mathbf{h}\right)$$

- Basis for all Euclidean isometries

# Reverse and Antireverse

- Reverse $\tilde{\mathbf{u}}$ multiplies vectors in reverse order
  - (with geometric product)

- Antireverse $\underset{\sim}{\mathbf{u}}$ multiplies antivectors in reverse order
  - (with geometric antiproduct)

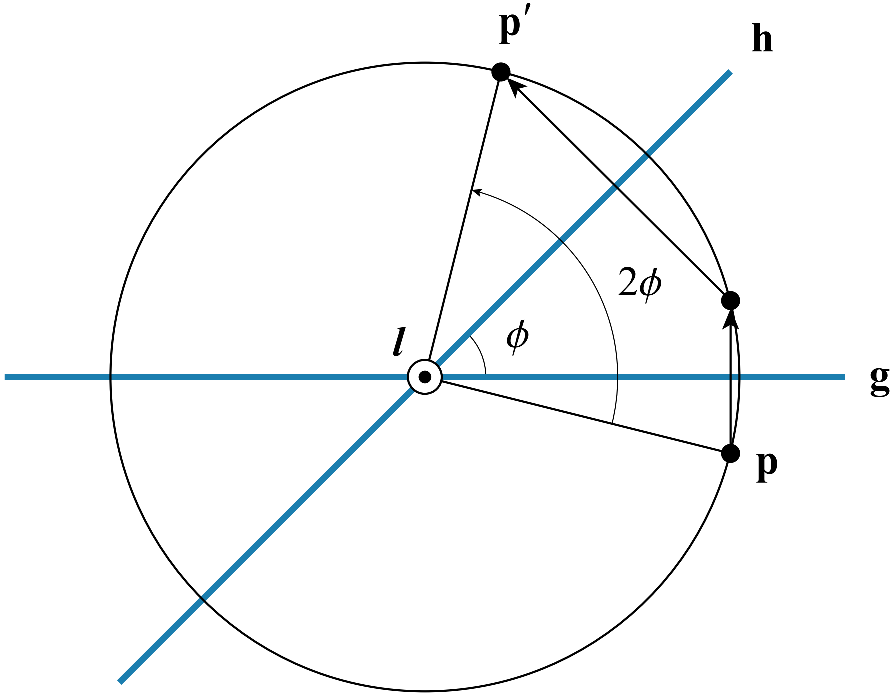
- Conjugate of quaternion is really a reverse operation

| $\mathbf{u}$ | $\mathbf{1}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_4$ | $\mathbf{e}_{41}$ | $\mathbf{e}_{42}$ | $\mathbf{e}_{43}$ | $\mathbf{e}_{23}$ | $\mathbf{e}_{31}$ | $\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\tilde{\mathbf{u}}$ | $\mathbf{1}$ | $\mathbf{e}_1$ | $\mathbf{e}_2$ | $\mathbf{e}_3$ | $\mathbf{e}_4$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | $-\mathbf{e}_{423}$ | $-\mathbf{e}_{431}$ | $-\mathbf{e}_{412}$ | $-\mathbf{e}_{321}$ | $\mathbb{1}$ |
| $\underset{\sim}{\mathbf{u}}$ | $\mathbf{1}$ | $-\mathbf{e}_1$ | $-\mathbf{e}_2$ | $-\mathbf{e}_3$ | $-\mathbf{e}_4$ | $-\mathbf{e}_{41}$ | $-\mathbf{e}_{42}$ | $-\mathbf{e}_{43}$ | $-\mathbf{e}_{23}$ | $-\mathbf{e}_{31}$ | $-\mathbf{e}_{12}$ | $\mathbf{e}_{423}$ | $\mathbf{e}_{431}$ | $\mathbf{e}_{412}$ | $\mathbf{e}_{321}$ | $\mathbb{1}$ |

# Rotation about a Line

- Let **g** and **h** be planes meeting at an angle $\phi$
- Reflection across **g** followed by **h** is rotation through $2\phi$ about line ***l*** where planes intersect

$$\boldsymbol{l} = \frac{\mathbf{h} \vee \mathbf{g}}{\left\| \mathbf{h} \vee \mathbf{g} \right\|_{\circ}}$$



# Rotation about a Line

- Planes multiply together under geometric antiproduct to form rotation operator **R**

$$\mathbf{p}' = \mathbf{h} \veedot \left( \mathbf{g} \veedot \mathbf{p} \veedot \mathbf{g} \right) \veedot \mathbf{h}$$

$$\mathbf{p}' = \mathbf{R} \veedot \mathbf{p} \veedot \underset{\sim}{\mathbf{R}}$$

$$\mathbf{R} = \mathbf{h} \veedot \mathbf{g}$$

# Rotation about a Line

- General form of rotation operator **R**:

$$\mathbf{R} = \boldsymbol{l}\sin\phi + \mathbb{1}\cos\phi$$

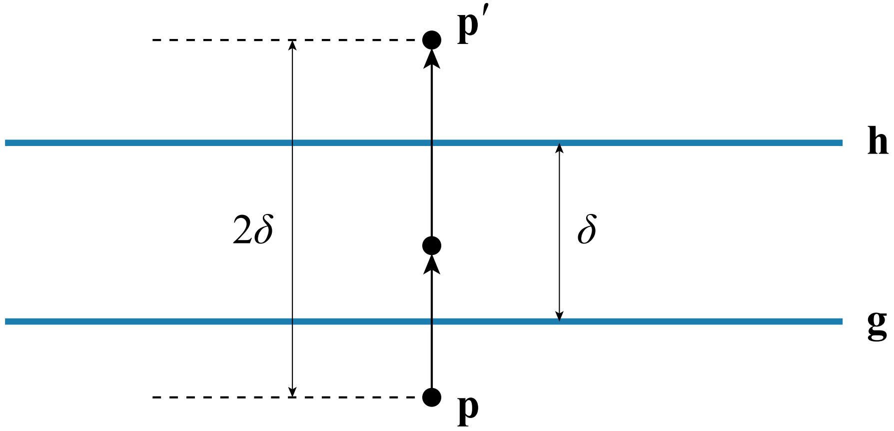
- Rotates through angle $2\phi$ about unitized line ***l***

$$\mathbf{u}' = \mathbf{R} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{R}}$$

- Rotates any geometry and even other operators

# Translation

- If planes **g** and **h** are parallel, result is a translation
- Translation goes along normal direction by twice the distance $\delta$ between the planes



# Translation

- General form of translation operator **T**:

$$\mathbf{T} = \tau_x \mathbf{e}_{23} + \tau_y \mathbf{e}_{31} + \tau_z \mathbf{e}_{12} + \mathbb{1}$$

- Translates by displacement vector 2t

$$\mathbf{u}' = \mathbf{T} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{T}}$$

- Translates any geometry and even other operators

# Euclidean Isometry Operators

- Sandwiches with geometric antiproduct perform Euclidean isometries
- Motor = MOtion operaTOR
- Flector = reFLECtion operaTOR

# Motor

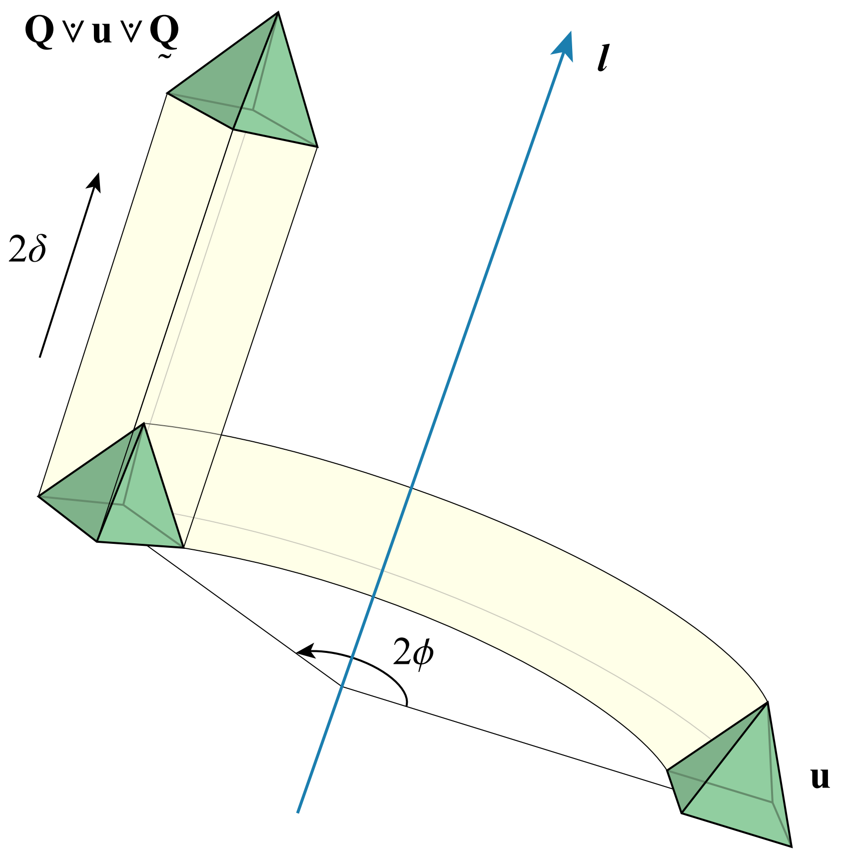
- General form of a motor:

$$\mathbf{Q} = \underbrace{Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1}}_{\text{Rotation Quaternion}} + \underbrace{Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}}_{\text{Moment and Displacement}}$$

- Performs any combination of rotations and translations

$$\mathbf{u}' = \mathbf{Q} \mathbin{\dot{\vee}} \mathbf{u} \mathbin{\dot{\vee}} \utilde{\mathbf{Q}}$$

# Motor



$$\mathbf{Q} = \exp_{\dot{\vee}}\left[\left(\delta\mathbf{1} + \varphi\mathbb{1}\right)\dot{\vee}\,\boldsymbol{l}\right] = \boldsymbol{l}\sin\varphi - \boldsymbol{l}^{\star}\delta\cos\varphi - \delta\sin\varphi + \mathbb{1}\cos\varphi$$

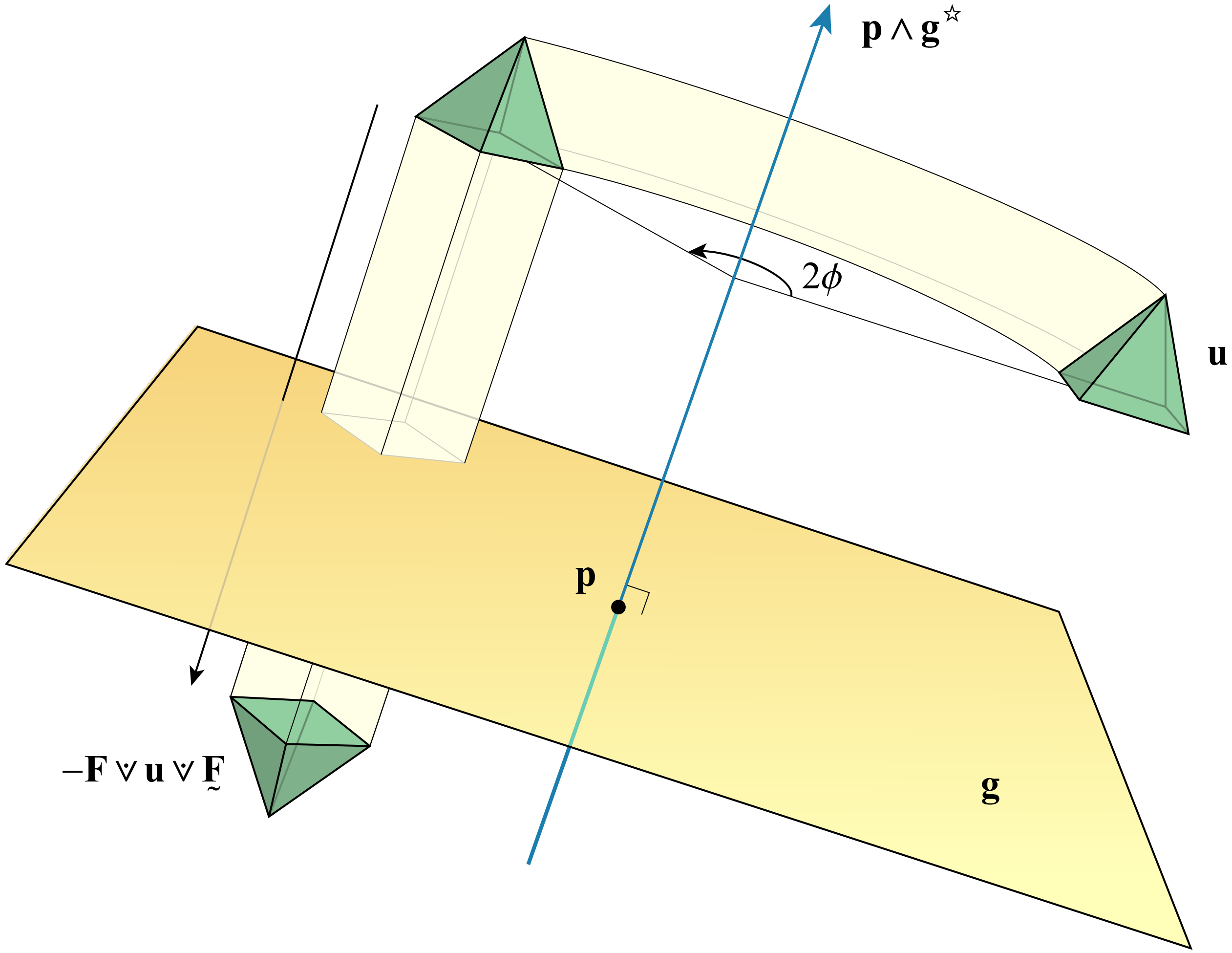
# Flector

- General form of a flector:

$$\mathbf{F} = \underbrace{F_{px}\,\mathbf{e}_1 + F_{py}\,\mathbf{e}_2 + F_{pz}\,\mathbf{e}_3 + F_{pw}\,\mathbf{e}_4}_{\text{Point}} + \underbrace{F_{gx}\,\mathbf{e}_{423} + F_{gy}\,\mathbf{e}_{431} + F_{gz}\,\mathbf{e}_{412} + F_{gw}\,\mathbf{e}_{321}}_{\text{Plane}}$$

- Performs any combination of rotoreflections

# Flector



$$\mathbf{F} = \mathbf{p}\sin\varphi + \mathbf{g}\cos\varphi$$

# Motor Parameterization

- A motion operator is parameterized by:
  - A unitized line ***l***
  - A rotation angle $\phi$
  - A displacement distance $\delta$

- Exponential with respect to geometric antiproduct:

$$\mathbf{Q} = \exp_{\dot{\vee}}\left[\left(\delta\mathbf{1} + \phi\mathbb{1}\right) \mathbin{\dot{\vee}} \boldsymbol{l}\right] = \boldsymbol{l}\sin\phi - \boldsymbol{l}^{\star}\delta\cos\phi - \delta\sin\phi + \mathbb{1}\cos\phi$$

- $\delta\mathbf{1} + \phi\mathbb{1}$ is *pitch* of screw transformation

Sec. 3.6.2

# Motor Parameterization

- Given arbitrary motor **Q**, can calculate parameters

$$\mathbf{Q} = Q_{vx}\,\mathbf{e}_{41} + Q_{vy}\,\mathbf{e}_{42} + Q_{vz}\,\mathbf{e}_{43} + Q_{vw}\,\mathbb{1} + Q_{mx}\,\mathbf{e}_{23} + Q_{my}\,\mathbf{e}_{31} + Q_{mz}\,\mathbf{e}_{12} + Q_{mw}\,\mathbf{1}$$

$$s = \sin\phi = \sqrt{1 - Q_{vw}^2} \qquad \delta = -\frac{Q_{mw}}{s} \qquad \phi = \tan^{-1}\left(\frac{s}{Q_{vw}}\right)$$

$$\boldsymbol{l}_{\mathbf{v}} = \frac{1}{s}\mathbf{Q}_{vxyz} \qquad \boldsymbol{l}_{\mathbf{m}} = \frac{1}{s}\left(\mathbf{Q}_{mxyz} + \frac{Q_{vw}Q_{mw}}{s^2}\mathbf{Q}_{vxyz}\right)$$

# Motor Interpolation

- To interpolate from motor $\mathbf{Q}_1$ to motor $\mathbf{Q}_2$, first calculate

$$\mathbf{Q}_0 = \mathbf{Q}_2 \mathbin{\dot{\vee}} \mathbf{Q}_1^{-1} = \mathbf{Q}_2 \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{Q}}_1$$

- Then calculate parameters $\boldsymbol{l}$, $\delta$, and $\phi$ for $\mathbf{Q}_0$

- Interpolate from identity $\mathbb{1}$ to $\mathbf{Q}_0$ with

$$\mathbf{Q}(t) = \exp_{\dot{\vee}}\left[t\left(\delta\mathbf{1} + \phi\mathbb{1}\right) \mathbin{\dot{\vee}} \boldsymbol{l}\right] = \boldsymbol{l}\sin(t\phi) - \boldsymbol{l}^{\star} t\delta\cos(t\phi) - t\delta\sin(t\phi) + \mathbb{1}\cos(t\phi)$$

- Finally, calculate $\mathbf{Q}(t) \mathbin{\dot{\vee}} \mathbf{Q}_1$

# Motor Interpolation

- That can be computationally expensive

- Approximate interpolation is often acceptable:

$$\mathbf{Q}\left(t\right)=\left(1-t\right)\mathbf{Q}_1+t\mathbf{Q}_2$$

- This needs to be unitized and constrained

$$\frac{\mathbf{Q}}{\left\|\mathbf{Q}_{\mathbf{v}}\right\|}\mathbin{\dot{\vee}}\left(-\frac{\mathbf{Q}_{\mathbf{v}}\cdot\mathbf{Q}_{\mathbf{m}}}{\mathbf{Q}_{\mathbf{v}}^2}\mathbf{1}+\mathbb{1}\right)=\frac{1}{\left\|\mathbf{Q}_{\mathbf{v}}\right\|}\left[\mathbf{Q}-\frac{\mathbf{Q}_{\mathbf{v}}\cdot\mathbf{Q}_{\mathbf{m}}}{\mathbf{Q}_{\mathbf{v}}^2}\left(Q_{vx}\mathbf{e}_{23}+Q_{vy}\mathbf{e}_{31}+Q_{vz}\mathbf{e}_{12}+Q_{vw}\right)\right]$$

# Square Root of Motor

- Special case of interpolation from $\mathbb{1}$ to **Q** when *t* = 1/2

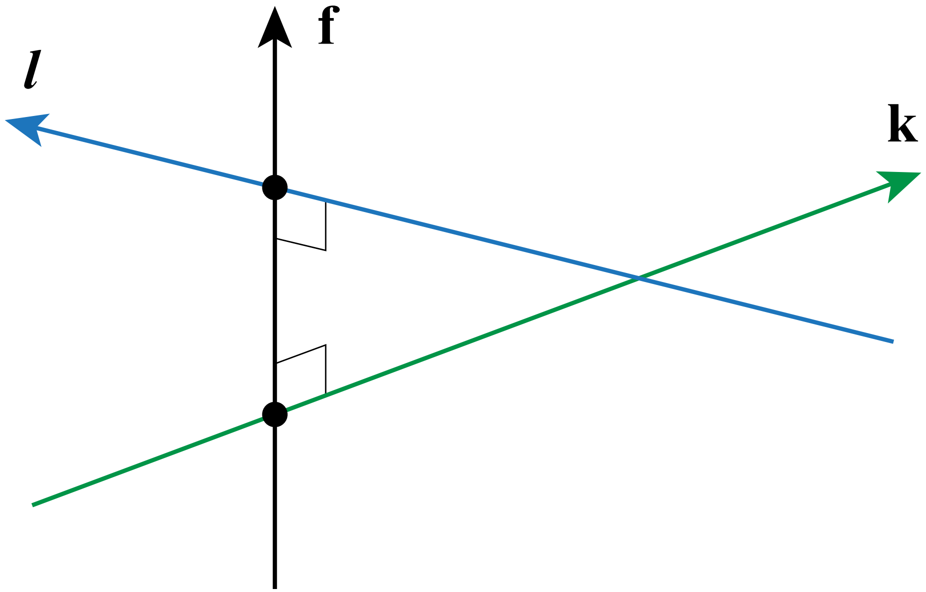
$$\sqrt[\dot{\vee}]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\sqrt{2 + 2Q_{\mathbb{1}}}} \mathbin{\dot{\vee}} \left( \mathbb{1} - \frac{Q_{\mathbf{1}}}{2 + 2Q_{\mathbb{1}}} \mathbf{1} \right)$$

- For simple motor (pure rotation or translation), this simplifies:

$$\sqrt[\dot{\vee}]{\mathbf{Q}} = \frac{\mathbf{Q} + \mathbb{1}}{\left\| \mathbf{Q} + \mathbb{1} \right\|_{\bigcirc}}$$

# Line to Line Motion

- Let **k** and ***l*** be lines separated by distance $\delta$ with angle $\phi$ between directions
- Operator $\boldsymbol{l} \mathbin{\dot{\vee}} \underset{\sim}{\mathbf{k}}$ rotates by $2\phi$ and translates by distance $2\delta$ about line **f** connecting closest points
- Square root of this operator transforms line **k** into line ***l***



Sec. 3.6.3

# Motor-Point Transformation

- 25 multiply-adds:

$$\mathbf{p}'_{xyz} = \mathbf{p}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a} - Q_{mw} p_w \mathbf{v}\right)$$

$$p'_w = p_w$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{p}_{xyz} + p_w \mathbf{m}$$

$$\mathbf{v} = \left(Q_{vx}, Q_{vy}, Q_{vz}\right)$$

$$\mathbf{m} = \left(Q_{mx}, Q_{my}, Q_{mz}\right)$$

- 3x4 matrix transformation only requires 12 multiply-adds, (or just 9 if $p_w = 1$)

# Motor-Line Transformation

- 54 multiply-adds:

$$\boldsymbol{l}'_{\mathbf{v}} = \boldsymbol{l}_{\mathbf{v}} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$\boldsymbol{l}'_{\mathbf{m}} = \boldsymbol{l}_{\mathbf{m}} + 2\left[Q_{mw}\mathbf{a} + Q_{vw}\left(\mathbf{b} + \mathbf{c}\right) + \mathbf{v} \times \left(\mathbf{b} + \mathbf{c}\right) + \mathbf{m} \times \mathbf{a}\right]$$

$$\mathbf{a} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{v}} \qquad \mathbf{b} = \mathbf{v} \times \boldsymbol{l}_{\mathbf{m}} \qquad \mathbf{c} = \mathbf{m} \times \boldsymbol{l}_{\mathbf{v}}$$

- 6x6 matrix transformation only requires 27 multiply-adds

# Motor-Plane Transformation

- 35 multiply-adds:

$$\mathbf{g}'_{xyz} = \mathbf{g}_{xyz} + 2\left(Q_{vw}\mathbf{a} + \mathbf{v} \times \mathbf{a}\right)$$

$$g'_w = g_w + 2\left[\left(\mathbf{m} \times \mathbf{g}_{xyz} + Q_{mw}\mathbf{g}_{xyz}\right) \cdot \mathbf{v} - Q_{vw}\left(\mathbf{m} \cdot \mathbf{g}_{xyz}\right)\right]$$

$$\mathbf{a} = \mathbf{v} \times \mathbf{g}_{xyz}$$

- 4x4 matrix transformation only requires 13 multiply-adds

# Motor to Matrix

$$\mathbf{A_Q} = \begin{bmatrix} 1-2\left(Q_{vy}^2+Q_{vz}^2\right) & 2Q_{vx}Q_{vy} & 2Q_{vz}Q_{vx} & 2\left(Q_{vy}Q_{mz}-Q_{vz}Q_{my}\right) \\ 2Q_{vx}Q_{vy} & 1-2\left(Q_{vz}^2+Q_{vx}^2\right) & 2Q_{vy}Q_{vz} & 2\left(Q_{vz}Q_{mx}-Q_{vx}Q_{mz}\right) \\ 2Q_{vz}Q_{vx} & 2Q_{vy}Q_{vz} & 1-2\left(Q_{vx}^2+Q_{vy}^2\right) & 2\left(Q_{vx}Q_{my}-Q_{vy}Q_{mx}\right) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{B_Q} = \begin{bmatrix} 0 & -2Q_{vz}Q_{vw} & 2Q_{vy}Q_{vw} & 2\left(Q_{vw}Q_{mx}-Q_{vx}Q_{mw}\right) \\ 2Q_{vz}Q_{vw} & 0 & -2Q_{vx}Q_{vw} & 2\left(Q_{vw}Q_{my}-Q_{vy}Q_{mw}\right) \\ -2Q_{vy}Q_{vw} & 2Q_{vx}Q_{vw} & 0 & 2\left(Q_{vw}Q_{mz}-Q_{vz}Q_{mw}\right) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{M_Q} = \mathbf{A_Q} + \mathbf{B_Q} \qquad \mathbf{M_Q^{-1}} = \mathbf{A_Q} - \mathbf{B_Q}$$

# Motor Composition

- 48 multiply-adds:

$$\begin{aligned}\mathbf{Q} \veedot \mathbf{R} = {} & \left(Q_{vw}R_{vx} + Q_{vx}R_{vw} + Q_{vy}R_{vz} - Q_{vz}R_{vy}\right)\mathbf{e}_{41} \\ & + \left(Q_{vw}R_{vy} - Q_{vx}R_{vz} + Q_{vy}R_{vw} + Q_{vz}R_{vx}\right)\mathbf{e}_{42} \\ & + \left(Q_{vw}R_{vz} + Q_{vx}R_{vy} - Q_{vy}R_{vx} + Q_{vz}R_{vw}\right)\mathbf{e}_{43} \\ & + \left(Q_{vw}R_{vw} - Q_{vx}R_{vx} - Q_{vy}R_{vy} - Q_{vz}R_{vz}\right)\mathbb{1} \\ & + \left(Q_{mw}R_{vx} + Q_{mx}R_{vw} + Q_{my}R_{vz} - Q_{mz}R_{vy} + Q_{vw}R_{mx} + Q_{vx}R_{mw} + Q_{vy}R_{mz} - Q_{vz}R_{my}\right)\mathbf{e}_{23} \\ & + \left(Q_{mw}R_{vy} - Q_{mx}R_{vz} + Q_{my}R_{vw} + Q_{mz}R_{vx} + Q_{vw}R_{my} - Q_{vx}R_{mz} + Q_{vy}R_{mw} + Q_{vz}R_{mx}\right)\mathbf{e}_{31} \\ & + \left(Q_{mw}R_{vz} + Q_{mx}R_{vy} - Q_{my}R_{vx} + Q_{mz}R_{vw} + Q_{vw}R_{mz} + Q_{vx}R_{my} - Q_{vy}R_{mx} + Q_{vz}R_{mw}\right)\mathbf{e}_{12} \\ & + \left(Q_{mw}R_{vw} - Q_{mx}R_{vx} - Q_{my}R_{vy} - Q_{mz}R_{vz} + Q_{vw}R_{mw} - Q_{vx}R_{mx} - Q_{vy}R_{my} - Q_{vz}R_{mz}\right)\mathbf{1}\end{aligned}$$

- Composition of equiv 3x4 matrices requires 33 multiply-adds

# Matrix Advantages

- Can represent more transformations
- Can read off origin and axis directions in transformed space
- Faster to transform objects
- Faster to compose

# Motor Advantages

- Smaller storage requirements
  - Usually 8 floats, but can reduce to 6

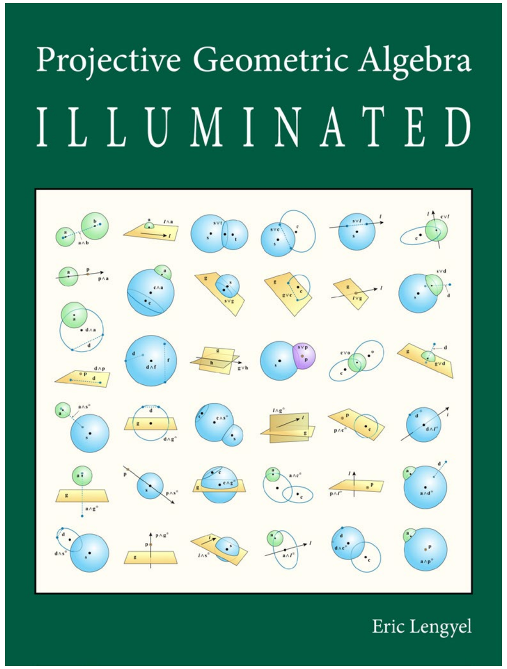
- Inversion is trivial
  - Just reverse, negating six bivector components

- Better parameterization

- Better interpolation properties

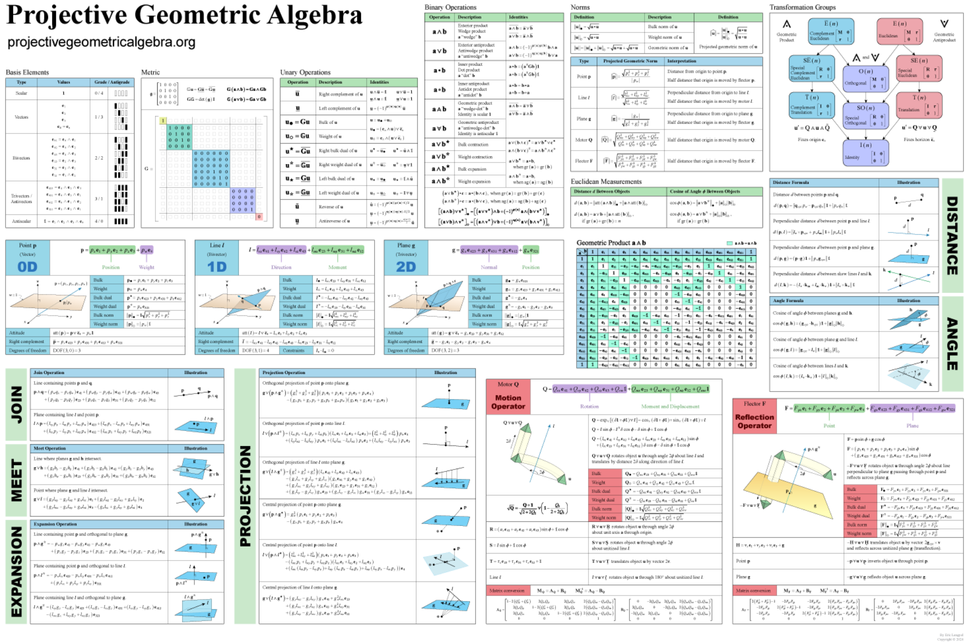
# References

- Projective Geometric Algebra Illuminated
- projectivegeometricalgebra.org



Projective Geometric Algebra
ILLUMINATED

Eric Lengyel



Projective Geometric Algebra
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